ECON 702 Macroeconomics I

Discussion Handout 1*

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Math review

- **Logarithms and Exponents.** Why should we review them? The manipulation of growth rates requires the use of logarithms and exponential functions.
  Exponentiation involves two numbers, the base b and the exponent n. Exponentiation corresponds to the operation
  $$b^n,$$
  that is multiplying the base b by itself n times.

  Logarithm is the inverse operation to exponentiation. It asks the question how many times do I have to multiply the base b by itself to obtain a given value x. The answer, call it z, therefore solves
  $$b^z = x,$$
  and z is called the logarithm of x with base b, written as
  $$\log_b(x).$$

  Natural logarithm uses as base the number $e = 2.71828$, so called Euler's number. The notation for the natural logarithm is $\log(x)$.

  Basic rules of Exponents and Logarithms. Exponents satisfy the following basic rules:
  $$b^{m+n} = b^m \cdot b^n$$
  $$(b^m)^n = b^{m \cdot n}$$
  $$(b \cdot c)^n = b^n \cdot c^n.$$

  For any numbers $x, y > 0$ logarithms satisfy the following rules:
  $$\log(x \cdot y) = \log(x) + \log(y)$$
  $$\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$$
  $$\log(x^n) = n \log(x),$$

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and for any base $b \neq e$ we can transform expressions into the natural logarithm by the formula

$$\log_b(x) = \frac{\log(x)}{\log(b)}.$$

- **First Order Taylor Series Expansions.**
  Consider an arbitrary function $f(x)$ that is differentiable. Then **the first order Taylor approximation of $f(x)$ around the point** x_0 is given by

  $$f(x) \approx f(x_0) + (x - x_0)f'(x_0).$$

  The idea is that close to x_0 ($x \to x_0$) any smooth (that is, differentiable) function is roughly linear in its argument x (with a slope of $f'(x_0)$).

Exercises on Growth Rates

We define the growth rate of a variable Y (say real GDP) from period t to $t+1$ as

$$g_{Y,t+1} = \frac{Y_{t+1} - Y_t}{Y_t}.$$

Assume discrete time for exercises 1-4 and continuous time for exercise 5.

1. Show that the growth rate of a variable Y_t is approximately equal to its log-difference from one period to another:

   $$g_{Y,t+1} \approx \log(Y_{t+1}) - \log(Y_t).$$

   Is this true for any size of the growth rate $g_{Y,t+1}$?

2. Suppose at period 0 GDP equals 100 and GDP grows at a constant rate of 3% per year. In how many years will GDP double? (Give an exact or an approximate answer.)

3. The growth rate of GDP, Y_t, equals 4%, and the growth rate of the population, N_t, equals 1%. Derive the formula for the growth rate of GDP per capita, $y_t = Y_t/N_t$, using the quotient (ratio) logarithm rule and the result from Exercise 1. Compute the growth rate of GDP per capita. Note that the formula gives a good approximation only when the growth rates are small.

4. Suppose that production follows a Cobb-Douglas technology: $Y_t = B_t K_t^{\alpha} L_t^{1-\alpha}$, where B_t denotes productivity in period t, K_t - capital in period t, and L_t - labor in period t. Derive a growth rate of Y_t as a function of growth rates of B_t, K_t, and L_t. What is the contribution of labor growth to GDP growth?

5. Show that in continuous time taking a logarithm of a function $x(t)$ of time and then taking the derivative with respect to time t gives the growth rate. Apply this approach to derive the growth rate of GDP per capita as a function of the growth rate of GDP and the growth rate of the population. What are the benefits of working in continuous time?