ECON 702 Macroeconomics I

Discussion Handout 10*

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A General Expression of Uncertainty

- In each period $t \geq 0$, a stochastic event $s_t \in S$ is realized.
- $s^t = (s_0, s_1, \ldots, s_t)$ denotes a history.
- Denote the unconditional probability distribution of histories at time t by $\pi_t(s^t)$.
- π's subscript t says that we consider the history over t periods. The superscript of s and the subscript of π are the same. We use the subscript of π to differentiate the probability of history s^t (denoted by $\pi_t(s^t)$) from the probability of an aggregate state for one period (denoted by $\pi(s_t)$).
- Conditional probability for two consecutive periods is denoted $Prob(s_{t+1} = s'|s_t = s) = \pi(s'|s)$.
- We can find $\pi_t(s^t)$ via recursion. Then

$$\pi_t(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})...\pi(s_1|s_0)\pi_0(s_0),$$

  where $\pi_0(s_0)$ denotes the initial ergodic distribution and equals 1 if s_0 is known.
- $\pi_t(s^t \mid s^\tau)$ denotes the probability of history s^t conditional on the realization of s^τ with $t > \tau$.
- We can express variables in the economy in terms of the stochastic state s^t.

Example: Markov chains

Markov Property: A stochastic process $\{s_t\}$ is said to have the Markov property if for all $k \geq 1$ and all t,

$$Prob(s_{t+1}|s_t, s_{t-1}, ..., s_{t-k}) = Prob(s_{t+1}|s_t)$$

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The Markov property can be rephrased as

$$\pi_t(s^{t+1}|s^t) = \pi(s_{t+1}|s_t).$$

A finite state Markov chain with n possible values $s_t \in \{x_1, ..., x_n\}$ is defined by an initial state or initial probabilities in combination with a $n \times n$ transition matrix P such that

$$P_{ij} = Prob(s_{t+1} = x_j | s_t = x_i)$$

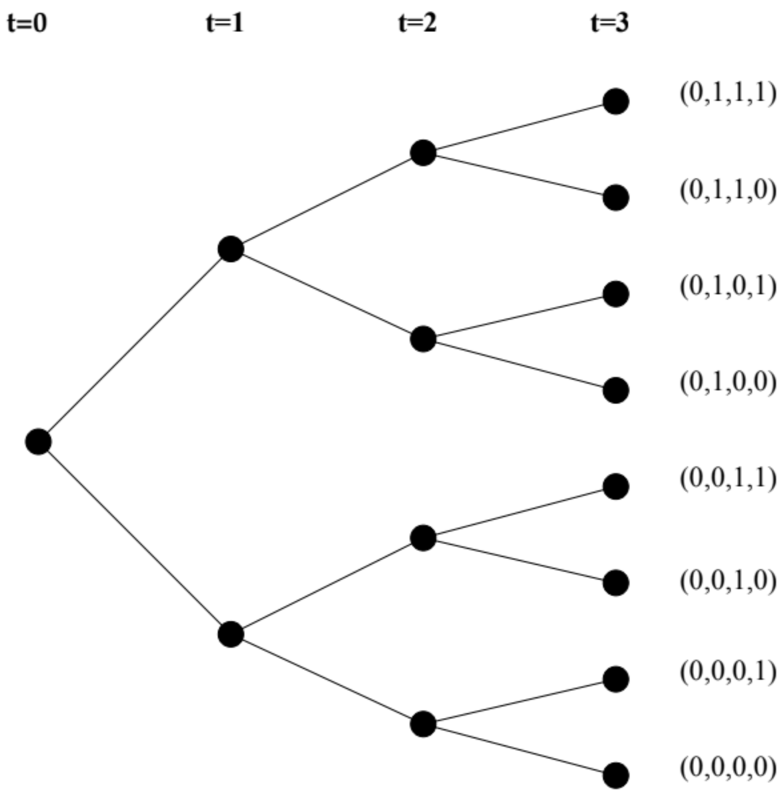
Thus, note that

$$\sum_{j=1}^{n} P_{ij} = 1.$$

When s follows the Markov process, the conditional probability $\pi_t(s^t|s^\tau)$ for $t > \tau$ depends only on the state s_τ at time τ and does not depend on the history before τ:

$$\pi_t(s^t|s^\tau) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})...\pi(s_{\tau+1}|s_\tau).$$

1. Suppose $s_t \in \{0, 1\}$ follows a Markov process, and $s_0 = 0$ (WLOG). Draw a tree to represent s_t in each $t = 0, 1, 2, 3$. Write s^t for each terminal node.



2. Suppose $Prob(s_{t+1} = 0|s_t = 0) = .5$ and $Prob(s_{t+1} = 0|s_t = 1) = .75$ for $t = 0, 1, 2$. Find $Prob(s_3 = 0)$ and $Prob(\sum_{t=0}^{3} s_t = 2)$.

Using the notation $P_{ij} = Prob(s_{t+1} = s_j | s_t = s_i)$, we can rewrite the given information as follows: $P_{00} = 0.5$ and $P_{10} = 0.75$. Note that this implies $P_{11} = 1 - P_{10} = 0.25, P_{01} = 1 - P_{00} = 1 - 0.5 = 0.5$.

There are 4 nodes in which $s_3 = 0$. We will iterate backwards to find the probability of being in each of these nodes.

$$Pr(s_3 = 0) = Pr(s_3 = 0|s_2 = 1) * Pr(s_2 = 1) + Pr(s_3 = 0|Pr(s_2 = 0) * Pr(s_2 = 0)$$

$$\begin{aligned} &= Pr(s_3 = 0|s_2 = 0)Pr(s_2 = 0|s_1 = 0)Pr(s_1 = 0|s_0 = 0) \\ &+Pr(s_3 = 0|s_2 = 1)Pr(s_2 = 1|s_1 = 0)Pr(s_1 = 0|s_0 = 0) \\ &+Pr(s_3 = 0|s_2 = 0)Pr(s_2 = 0|s_1 = 1)Pr(s_1 = 1|s_0 = 0) \\ &+Pr(s_3 = 0|s_2 = 1)Pr(s_2 = 1|s_1 = 1)Pr(s_1 = 1|s_0 = 0) \end{aligned}$$

$$= P_{00}P_{00}P_{00} + P_{10}P_{01}P_{00} + P_{00}P_{10}P_{01} + P_{10}P_{11}P_{01}$$
$$= P_{00}^3 + P_{01}P_{10}(2P_{00} + P_{11}) = .5^3 + .75 * .5(1 + .25) = .125 + .46875 = .59375$$

There are 3 nodes which have $Prob(\sum_{t=0}^{3} s_t = 2)$. Those are (0,0,1,1), (0,1, 0, 1), and (0, 1, 1, 0).

$$Prob(\sum_{t=0}^{3} s_t = 2) = P_{11}P_{01}P_{00}+P_{01}P_{10}P_{01}+P_{01}P_{11}P_{10} = .25*.5^2+.5^2*.75+.75*.25*.5 = .34375$$

3. Is an AR(1) process a Markov process?

   Yes, an AR(1) process is a Markov process.

   $$x_{t+1} = \rho x_t + \sigma * \varepsilon_t$$

   where ε_t are iid and mean 0, typically $\varepsilon_t \sim iidN(0, 1)$.

   We can see that the conditional distribution $f(x_{t+1}|x_t) \sim N(\rho x_t, \sigma)$ is free of $x_{t-1}, x_{t-2}, ...$

   Thus, the AR(1) process is a Markov process.

Neoclassical Growth With Uncertainty

Consider the typical Neoclassical growth model, but with an uncertain aggregate state in each period, with histories indexed by s^t with probabilities $\pi_t(s^t)$. The discount rate β, the rate of depreciation of capital δ, and the representative household's utility function $u(c)$ remain constant, but other parameters and variables depend on the state s^t. For generality, we write the production technology $F(s^t, K_t(s^{t-1}), L_t(s^t))$ which is CRS. Households exogenously supply labor $L_t(s^t)$ and consume the final good $C_t(s^t)$.

1. Discuss some potential economic meanings for the aggregate state.

   There are many potential meanings for the aggregate state. One is TFP shocks, which we have discussed. Others include changes in labor endowment, monetary shocks, fiscal shocks, etc. All of these are special cases of the flexible model we have written down. This includes the case of the AR(1) process of TFP shocks we studied in the business cycle model.

2. Write the social planner's problem from time 0 in general.

$$\max_{C_t(s^t),K_t(s^t)} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t(s^t)) = \sum_{t=0}^{\infty}\sum_{s^t} \pi_t(s^t)\beta^t u(C_t(s^t))$$

   subject to

$$C(s^t) + K_{t+1}(s^t) = F(s^t, K_t(s^{t-1}), L(s^t)) + (1-\delta)K_t(s^{t-1}) \quad \forall t, s^t \supset s^{t-1}$$

   Note that we are looking for paths of choice variables over all periods. But we only need the budget constraint to hold for histories s^t which contain s^{t-1}.

3. Formulate the Lagrangian of the social planner and derive the Euler equation.

$$\begin{aligned}&\mathcal{L} = \\ &\sum_{t=0}^{\infty}\sum_{s^t} \pi_t(s^t)\beta^t u(C_t(s^t)) + \\ &\sum_{s^t \supset s^{t-1}} \lambda_t(s^t)\left[F(s^t, K_t(s^{t-1}), L_t(s^t)) + (1-\delta)K_t(s^{t-1}) - C(s^t) - K_{t+1}(s^t)\right]\end{aligned}$$

   Note that the Lagrange multiplier $\lambda_t(s^t)$ depends on the state, and we have included the probablity $\pi_t(s^t)$ in the multiplier. The initial Lagrangian could be written as

$$\begin{aligned}&\mathcal{L} = \\ &\sum_{t=0}^{\infty}\sum_{s^t}(\pi_t(s^t)\beta^t u(C_t(s^t)) + \\ &\sum_{s^t \supset s^{t+1}} \pi_t(s^t)\tilde{\lambda}_t(s^t)[F(s^t, K_t(s^{t-1}), L_t(s^t)) + \\ &+ (1-\delta)K_t(s^{t-1}) - C(s^t) - K_{t+1}(s^t)])\end{aligned}$$

   and then we redefine the Lagrange multiplier as follows: $\lambda(s^t) = \pi_t(s^t)\tilde{\lambda}_t(s^t)$. We can do this in general, as it's just a rescaling of our multipliers. To see why this is true, multiply the budget constraint in each period by $\frac{1}{\pi_t(s^t)}$ and formulate the Lagrangian without incorporating the probability into the multipliers. You should get the same thing.

$$\begin{aligned}\text{FOC} \quad [C_t(s^t)]:& \quad \pi_t(s^t)\beta^t u'(C_t(s^t)) = \lambda_t(s^t) \\ [C_{t+1}(s^{t+1})]:& \quad \pi_{t+1}(s^{t+1})\beta^{t+1}u'(C_{t+1}(s^{t+1})) = \lambda_{t+1}(s^{t+1}) \\ [K_{t+1}(s^t)]:& \quad \lambda_t(s^t) = \sum_{s^{t+1}\supset s^t} \lambda_{t+1}(s^{t+1})\left[F_K(s^{t+1}, K_{t+1}(s^t), L_{t+1}(s^{t+1})) + (1-\delta)\right]\end{aligned}$$

Combining FOCs, we get the Euler equation:

$$\pi_t(s^t)u'(C_t(s^t)) = \sum_{s^{t+1} \supset s^t} \pi_{t+1}(s^{t+1})\beta u'(C_{t+1}(s^{t+1})) \left[F_K(s^{t+1}, K_{t+1}(s^t), L_{t+1}(s^{t+1})) + (1-\delta)\right]$$

or

$$\pi_t(s^t)u'(C_t(s^t)) = \sum_{s^{t+1}} \pi_{t+1}(s^t, s^{t+1})\beta u'(C_{t+1}(s^{t+1})) \left[F_K(s^{t+1}, K_{t+1}(s^t), L_{t+1}(s^{t+1})) + (1-\delta)\right]$$

Since $s^{t+1} = \{s^t, s_{t+1}\}$ (if $s^t \subset s^{t+1}$), $\frac{\pi_{t+1}(s^t, s^{t+1})}{\pi_t(s^t)}$ can be rewritten as conditional probability

$$\frac{\pi_{t+1}(s^{t+1}, s^t)}{\pi_t(s^t)} = \pi_{t+1}(s^{t+1}|s^t).$$

Thus,

$$u'(C_t(s^t)) = \sum_{s^{t+1}} \pi_{t+1}(s^{t+1}|s^t)\beta u'(C_{t+1}(s^{t+1})) \left[F_K(s^{t+1}, K_{t+1}(s^t), L_{t+1}(s^{t+1})) + (1-\delta)\right]$$

Then the final Euler equations written with expectation operator is

$$u'(C_t(s^t)) = \beta \mathbb{E}_t \left[u'(C_{t+1}(s^{t+1})) \left(F_K(s^{t+1}, K_{t+1}(s^t), L_{t+1}(s^{t+1})) + (1-\delta)\right)\right]$$

Where we take the perspective of time t since s^t is observed at that point.