ECON 702 Macroeconomics I

Discussion Handout 11*

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1 Money in Utility

Consider an infinite-horizon model economy in which households derive utility from consumption, c_t, and real cash holdings, $\frac{m_t}{p_t}$, as some goods can only be bought using cash. Households supply labor, l_t, at price w_t in a competitive labor market and get disutility from work. Besides, they rent out capital, k_t, to the firms at price μ_t in a competitive capital market. Each period using labor income, households can consume, c_t, invest in capital (save), k_{t+1}, keep cash, m_{t+1}, and buy nominal bonds, b_{t+1}. Nominal bonds cost 1 nominal unit in period t and pay R_{t+1} nominal units in period $t+1$. Household preferences are given by:

$$U\left(c_t, l_t, \frac{m_t}{p_t}\right) = \log\left(c_t - \psi\frac{l_t^{1+\eta}}{1+\eta}\right) + \xi\log\left(\frac{m_t}{p_t}\right).$$

Firms produce using Cobb-Douglas technology:

$$Y_t = F(k_t, l_t) = k_t^{\alpha}(A_t l_t)^{1-\alpha}.$$

Assume that technology A_t is deterministic and capital fully depreciates during production in each period, $\delta = 1$, and households discount their future with the discount factor $\beta \in (0,1)$.

The government has a money supply rule:

$$\begin{aligned} m_t &= m e^{\hat{m}_t} \\ \hat{m}_t &= \rho_m \hat{m}_{t-1} + \sigma_m \varepsilon_{m,t-1}, \end{aligned}$$

where m is the mean of the money supply and $\hat{m}_t$ is a log-deviation that follows AR-1 process. The government has to balance its budget:

$$T_t + R_t\frac{b_t}{p_t} = \frac{1}{p_t}(m_{t+1} - m_t) + \frac{b_{t+1}}{p_t}.$$

1. Write the household problem and derive the first-order conditions.

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2. Using the first-order condition derive the Euler equations for money holdings and nominal bonds. Derive the non-arbitrage condition between money holdings and nominal bond holdings. Provide an interpretation.

3. If the nominal gross return on bonds, R_{t+1}, equals 1, does the non-arbitrage condition hold? How does the model parameter ξ affect the answer? Provide an interpretation.

4. If $\xi > 0$, under which condition for R_{t+1} agents hold both types of nominal assets in the equilibrium?

5. Suppose that the decision rule for capital and consumption is $k_{t+1} = \alpha\beta y_t$ and $c_t = (1-\alpha\beta)y_t$ (Why is it true?). Using the Euler equation for the cash holdings, derive the optimality condition for the price, p_t, as a function of y_t, m_t, and the model parameters.

Solution.

1. The environment is stochastic due to the monetary shocks, implying that households maximize expected lifetime utility and the choices are dependent on the realization of shocks. We will use the notation s^t to denote the vector of the realization of m_t in periods $0, 1, ..., t$. Note that in the lecture notes this notation is dropped. In the handout, we will drop the notation only for prices first. The household problem is:

$$\max_{\{c_t(s^t), l_t(s^t), k_{t+1}(s^t), m_{t+1}(s^t), b_{t+1}(s^t)\}} \mathbf{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\log\left(c_t(s^t) - \psi \frac{l_t(s^t)^{1+\eta}}{1+\eta}\right) + \xi \log\left(\frac{m_t(s^{t-1})}{p_t}\right)\right)$$

$$\text{s.t. } c_t(s^t) + k_{t+1}(s^t) + \frac{m_{t+1}(s^t)}{p_t} + \frac{b_{t+1}(s^t)}{p_t} =$$
$$= w_t l_t(s^t) + \mu_t k_t(s^{t-1}) + (1-\delta)k_t(s^{t-1}) + \frac{m_t(s^{t-1})}{p_t} + R_t \frac{b_t(s^{t-1})}{p_t} + T_t \quad \forall t, \forall s^t$$

To be more precise, all prices also depend on s^t. Note that the budget constraint is written in the real terms (units of goods). The lifetime utility function is expected. If we assume that we know the probabilities of the realization of each state and history, the Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi_t(s^t)\left(\log\left(c_t(s^t) - \psi\frac{l_t(s^t)^{1+\eta}}{1+\eta}\right) + \xi\log\left(\frac{m_t(s^{t-1})}{p_t}\right)\right) +$$
$$+ \sum_{t=0}^{\infty}\sum_{s^t} \lambda_t(s^t)\pi_t(s^t)(w_t l_t(s^t) + \mu_t k_t(s^{t-1}) + (1-\delta)k_t(s^{t-1}) + \frac{m_t(s^{t-1})}{p_t} + R_t\frac{b_t(s^{t-1})}{p_t} + T_t -$$
$$- c_t(s^t) - k_{t+1}(s^t) - \frac{m_{t+1}(s^t)}{p_t} - \frac{b_{t+1}(s^t)}{p_t})$$

Let us keep $\lambda_t(s^t)\pi_t(s^t)$ without redefining the Lagrange multiplier. Next, we derive the first-order conditions for each period and each possible history. For simplicity, for a given history s^t, we make decisions as if we were in period t, implying that $\pi_t(s^t) = 1$ and history for the next period s^{t+1} should include a given history s^t. Hence, the first-order conditions are

$$
\begin{aligned}
&[c_t(s^t)]: \quad \frac{\beta^t}{c_t(s^t) - \psi \frac{l_t(s^t)^{1+\eta}}{1+\eta}} = \lambda_t(s^t) \\
&[c_{t+1}(s^{t+1}]: \quad \frac{\beta^{t+1}\pi_{t+1}(s^{t+1})}{c_{t+1}(s^{t+1}) - \psi \frac{l_{t+1}(s^{t+1})^{1+\eta}}{1+\eta}} = \pi_{t+1}(s^{t+1})\lambda_{t+1}(s^{t+1}) \quad \forall s^{t+1} \text{ such that } s^t \subset s^{t+1} \\
&[l_t(s^t)]: \quad \frac{\beta^t}{c_t(s^t) - \psi \frac{l_t(s^t)^{1+\eta}}{1+\eta}} \psi l_t^\eta = \lambda_t(s^t) w_t \\
&[k_{t+1}(s^t)]: \quad \lambda_t(s^t) = \pi_{t+1}(s^{t+1})\lambda_{t+1}(s^{t+1})\left(\mu_{t+1} + 1 - \delta\right) \quad \forall s^{t+1} \text{ such that } s^t \subset s^{t+1} \\
&[m_{t+1}(s^t)]: \quad \beta^{t+1}\xi \frac{1}{\frac{m_{t+1}(s^t)}{p_{t+1}}} \frac{1}{p_{t+1}} + \lambda_{t+1}(s^{t+1})\frac{1}{p_{t+1}} - \lambda_t(s^t)\frac{1}{p_t} = 0 \rightarrow \\
&\quad \lambda_t(s^t) = \frac{p_t}{p_{t+1}}\left(\beta^{t+1}\frac{\xi}{\frac{m_{t+1}(s^t)}{p_{t+1}}} + \lambda_{t+1}(s^{t+1})\right) \quad \forall s^{t+1} \text{ such that } s^t \subset s^{t+1} \\
&[b_{t+1}(s^t)]: \quad \lambda_t(s^t)\frac{1}{p_t} = \lambda_{t+1}(s^{t+1})\pi_{t+1}(s^{t+1})\frac{R_{t+1}}{p_{t+1}}
\end{aligned}
$$

Note that the FOCs w.r.t. $c_{t+1}(s^{t+1}), k_{t+1}(s^t), m_{t+1}(s^t), b_{t+1}(s^t)$ should be satisfied for any history s^{t+1} which includes the history s^t. These three conditions can be expressed using expectations formed with information in period t. Let us sum the FOC w.r.t.$c_{t+1}(s^{t+1})$ over all s^{t+1} such that $s^t \subset s^{t+1}$. Then:

$$
\begin{aligned}
&[c_{t+1}(s^{t+1})]: \quad \sum_{s^{t+1}:s^t \subset s^{t+1}} \frac{\beta^{t+1}\pi_{t+1}(s^{t+1})}{c_{t+1}(s^{t+1}) - \psi \frac{l_{t+1}(s^{t+1})^{1+\eta}}{1+\eta}} = \sum_{s^{t+1}:s^t \subset s^{t+1}} \pi_{t+1}(s^{t+1})\lambda_{t+1}(s^{t+1}) \quad \forall s^{t+1} \\
&\beta^{t+1}\mathbf{E}_t\left[\frac{1}{c_{t+1}(s^{t+1}) - \psi \frac{l_{t+1}(s^{t+1})^{1+\eta}}{1+\eta}}\right] = \mathbf{E}_t[\lambda_{t+1}(s^{t+1})].
\end{aligned}
$$

Note this is equivalent to the FOC w.r.t. c_{t+1} you saw on the slides (in which notation of the history is dropped):

$$
\beta^{t+1}\mathbf{E}_t\left[u_1\left(c_{t+1}, l_{t+1}, \frac{m_{t+1}}{p_{t+1}}\right)\right] = \mathbf{E}_t[\lambda_{t+1}].
$$

Using the same manipulations we can rewrite the FOCs w.r.t. $k_{t+1}(s^t), m_{t+1}(s^t), b_{t+1}(s^t)$ as follows:

$$
\begin{aligned}
&[k_{t+1}(s^t)]: \quad \lambda_t(s^t) = \mathbf{E}_t\left[\lambda_{t+1}(s^{t+1})(\mu_{t+1} + 1 - \delta)\right] \\
&[m_{t+1}(s^t)]: \quad \lambda_t(s^t) = p_t\mathbf{E}_t\left[\frac{1}{p_{t+1}}\left(\beta^{t+1}\xi\frac{p_{t+1}}{m_{t+1}} + \lambda_{t+1}(s^{t+1})\right)\right] \\
&[b_{t+1}(s^t)]: \quad \lambda_t(s^t) = p_t\mathbf{E}_t\left[\lambda_{t+1}(s^{t+1})\frac{R_{t+1}}{p_{t+1}}\right].
\end{aligned}
$$

2. For further questions we drop notations for history. To derive the Euler equations for real cash holdings and for nominal bonds we use the FOCs w.r.t. c_t, c_{t+1}, m_{t+1}, b_{t+1}. The Euler equation for money holdings is

$$
\frac{1}{c_t - \psi\frac{l_t^{1+\eta}}{1+\eta}} = \mathbf{E}_t\left[\frac{p_t}{p_{t+1}}\beta\left(\xi\frac{p_{t+1}}{m_{t+1}} + \frac{1}{c_{t+1} - \psi\frac{l_{t+1}^{1+\eta}}{1+\eta}}\right)\right].
$$

The Euler equation for nominal bonds is

$$\frac{1}{c_t - \psi\frac{l_t^{1+\eta}}{1+\eta}} = \mathbf{E}_t\left[\frac{p_t}{p_{t+1}}\frac{\beta}{c_{t+1} - \psi\frac{l_{t+1}^{1+\eta}}{1+\eta}}R_{t+1}\right].$$

Combining the two Euler equations we obtain the non-arbitrage condition:

$$\mathbf{E}_t\left[\frac{p_t}{p_{t+1}}\beta\left(\xi\frac{p_{t+1}}{m_{t+1}} + \frac{1}{c_{t+1} - \psi\frac{l_{t+1}^{1+\eta}}{1+\eta}}\right)\right] = \mathbf{E}_t\left[\frac{p_t}{p_{t+1}}\frac{\beta}{c_{t+1} - \psi\frac{l_{t+1}^{1+\eta}}{1+\eta}}R_{t+1}\right].$$

The condition says that at the margin agents should be indifferent between holding an extra unit of money and one extra unit of nominal bond.

3. If $R_{t+1} = 1$, the non-arbitrage condition becomes

$$\mathbf{E}_t\left[\frac{p_t}{p_{t+1}}\beta\left(\xi\frac{p_{t+1}}{m_{t+1}} + \frac{1}{c_{t+1} - \psi\frac{l_{t+1}^{1+\eta}}{1+\eta}}\right)\right] = \mathbf{E}_t\left[\frac{p_t}{p_{t+1}}\frac{\beta}{c_{t+1} - \psi\frac{l_{t+1}^{1+\eta}}{1+\eta}}\right].$$

Note that the left-hand side can be written as follows:

$$\mathbf{E}_t\left[\frac{p_t}{p_{t+1}}\beta\xi\frac{p_{t+1}}{m_{t+1}} + \frac{p_t}{p_{t+1}}\frac{\beta}{c_{t+1} - \psi\frac{l_{t+1}^{1+\eta}}{1+\eta}}\right] = \mathbf{E}_t\left[\frac{p_t}{p_{t+1}}\beta\xi\frac{p_{t+1}}{m_{t+1}}\right] + \mathbf{E}_t\left[\frac{p_t}{p_{t+1}}\frac{\beta}{c_{t+1} - \psi\frac{l_{t+1}^{1+\eta}}{1+\eta}}\right].$$

It implies that the non-arbitrage condition is

$$\mathbf{E}_t\left[\frac{p_t}{p_{t+1}}\beta\xi\frac{p_{t+1}}{m_{t+1}}\right] = 0.$$

If $\xi \neq 0$, then the non-arbitrage condition cannot hold. Hence if $R_{t+1} = 1$ and $\xi > 0$, then money will be chosen over the nominal bonds. Why? Money and bonds bring the same nominal return of one, but money brings utility to the agents. If $R_{t+1} = 1$ and $\xi = 0$, then the households are always indifferent between money and nominal bonds. However, if $R_{t+1} > 1$ and $\xi = 0$, then money are never chosen. And if $R_{t+1} < 1$ and $\xi = 0$, the nominal bonds are never chosen.

4. If $\xi > 0$, in an equilibrium in which agents hold two types of nominal assets, R_{t+1} has to be larger than 1 (since the nominal bonds do not bring utility, while the money brings). We provide the loosest lower bound for R_{t+1}. Besides, you can think about whether there is an upper bound for R_{t+1}.

5. Recall that the Euler equation for the cash holdings is

$$\frac{1}{c_t - \psi\frac{l_t^{1+\eta}}{1+\eta}} = \mathbf{E}_t\left[\frac{p_t}{p_{t+1}}\beta\left(\xi\frac{p_{t+1}}{m_{t+1}} + \frac{1}{c_{t+1} - \psi\frac{l_{t+1}^{1+\eta}}{1+\eta}}\right)\right].$$

Our goal is to express $c_t - \psi\frac{l_t^{1+\eta}}{1+\eta}$ as a function of y_t. We can guess and verify that $c_t = (1-\alpha\beta)y_t$. It will work in the described model setup since $\delta = 1$. From the FOCs w.r.t. c_t and l_t we can also find optimality condition for labor supply:

$$\psi l_t^{\eta} = w_t,$$

where w_t has to be equal MPL because firms behave optimally in competitive markets

$$w_t = (1-\alpha)k_t^{\alpha}(A_t l_t)^{1-\alpha}\frac{1}{l_t} = (1-\alpha)\frac{y_t}{l_t}$$

Hence,

$$\psi l_t^{\eta} = w_t = (1-\alpha)\frac{y_t}{l_t}$$
$$\psi l_t^{1+\eta} = (1-\alpha)y_t.$$

$$c_t - \psi\frac{l_t^{1+\eta}}{1+\eta} = (1-\alpha\beta)y_t - \frac{1-\alpha}{1+\eta}y_t = \left(1-\alpha\beta-\frac{1-\alpha}{1+\eta}\right)y_t.$$

Let Φ denotes $1-\alpha\beta-\frac{1-\alpha}{1+\eta}$. Then, the Euler equation becomes

$$\frac{1}{\Phi y_t} = \mathbf{E}_t\left[\frac{p_t}{p_{t+1}}\beta\left(\xi\frac{p_{t+1}}{m_{t+1}} + \frac{1}{\Phi y_{t+1}}\right)\right].$$

To derive the optimality condition for p_t we need to iterate the equation forward with respect to $\frac{1}{\Phi y_t}$:

$$\begin{aligned}
\frac{1}{\Phi y_t} &= \mathbf{E}_t\left[\frac{p_t}{p_{t+1}}\beta\left(\xi\frac{p_{t+1}}{m_{t+1}} + \mathbf{E}_{t+1}\left[\frac{p_{t+1}}{p_{t+2}}\beta\left(\xi\frac{p_{t+2}}{m_{t+2}} + \frac{1}{\Phi y_{t+2}}\right)\right]\right) = \right.\\
&= p_t\mathbf{E}_t\left[\beta\xi\frac{1}{m_{t+1}} + \beta\mathbf{E}_{t+1}\left[\frac{1}{p_{t+2}}\beta\left(\xi\frac{p_{t+2}}{m_{t+2}} + \mathbf{E}_{t+2}\left[\frac{p_{t+2}}{p_{t+3}}\beta\left(\xi\frac{p_{t+3}}{m_{t+3}} + \frac{1}{\Phi y_{t+3}}\right)\right]\right)\right]\right] = \\
&= p_t\mathbf{E}_t\left[\beta\xi\frac{1}{m_{t+1}} + \beta^2\xi\mathbf{E}_{t+1}\left[\frac{1}{m_{t+1}}\right] + \beta^3\xi\mathbf{E}_{t+1}\left[\mathbf{E}_{t+2}\frac{1}{m_{t+3}}\right] + \beta^2\mathbf{E}_{t+1}\left[\mathbf{E}_{t+2}\left[\frac{1}{p_{t+3}}\frac{1}{\Phi y_{t+3}}\right]\right]\right] = \\
&= \ldots = p_t\xi\left[\beta\mathbf{E}_t\left[\frac{1}{m_{t+1}}\right] + \beta^2\mathbf{E}_t\left[\frac{1}{m_{t+2}}\right] + \beta^3\mathbf{E}_t\left[\frac{1}{m_{t+3}}\right] + \ldots\right] = p_t\xi\sum_{j=1}^{\infty}\beta^j\mathbf{E}_t\left[\frac{1}{m_{t+j}}\right].
\end{aligned}$$

Hence,

$$p_t = \frac{1}{\xi\Phi y_t}\left(\sum_{j=1}^{\infty}\beta^j\mathbf{E}_t\left[\frac{1}{m_{t+j}}\right]\right)^{-1}.$$