ECON 702 Macroeconomics I

Discussion Handout 1*

Solutions

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Math review

- **Logarithms and Exponents.** Why should we review them? The manipulation of growth rates requires the use of logarithms and exponential functions.
  Exponentiation involves two numbers, the base b and the exponent n. Exponentiation corresponds to the operation
  $$b^n,$$
  that is multiplying the base b by itself n times.

  Logarithm is the inverse operation to exponentiation. It asks the question how many times do I have to multiply the base b by itself to obtain a given value x. The answer, call it z, therefore solves
  $$b^z = x,$$
  and z is called the logarithm of x with base b, written as
  $$\log_b(x).$$

  Natural logarithm uses as base the number $e = 2.71828$, so called Euler's number. The notation for the natural logarithm is $\log(x)$.

  Basic rules of Exponents and Logarithms. Exponents satisfy the following basic rules:
  $$\begin{aligned} b^{m+n} &= b^m \cdot b^n \\ (b^m)^n &= b^{m \cdot n} \\ (b \cdot c)^n &= b^n \cdot c^n. \end{aligned}$$
  For any numbers $x, y > 0$ logarithms satisfy the following rules:
  $$\begin{aligned} \log(x \cdot y) &= \log(x) + \log(y) \\ \log\left(\frac{x}{y}\right) &= \log(x) - \log(y) \\ \log(x^n) &= n \log(x), \end{aligned}$$

*Teaching Assistants: Anna Lukianova (Email:lukianova@wisc.edu) and John Ryan (Email: john.p.ryan@wisc.edu). Based on the lecture notes by Jesus Fernandez-Villaverde and Dirk Krueger.

and for any base $b \neq e$ we can transform expressions into the natural logarithm by the formula

$$\log_b(x) = \frac{\log(x)}{\log(b)}.$$

- **First Order Taylor Series Expansions.**
  Consider an arbitrary function $f(x)$ that is differentiable. Then **the first order Taylor approximation of $f(x)$ around the point** x_0 is given by

  $$f(x) \approx f(x_0) + (x - x_0)f'(x_0).$$

  The idea is that close to x_0 ($x \to x_0$) any smooth (that is, differentiable) function is roughly linear in its argument x (with a slope of $f'(x_0)$).

Exercises on Growth Rates

We define the growth rate of a variable Y (say real GDP) from period t to $t+1$ as

$$g_{Y,t+1} = \frac{Y_{t+1} - Y_t}{Y_t}.$$

Assume discrete time for exercises 1-4 and continuous time for exercise 5.

1. Show that the growth rate of a variable Y_t is approximately equal to its log-difference from one period to another:

   $$g_{Y,t+1} \approx \log(Y_{t+1}) - \log(Y_t).$$

   Is this true for any size of the growth rate $g_{Y,t+1}$?

   Solution. The definition of the growth rate implies

   $$\begin{aligned} g_{Y,t+1}Y_t &= Y_{t+1} - Y_t \\ (1 + g_{Y,t+1})Y_t &= Y_{t+1}. \end{aligned}$$

   Now take the natural logarithms on both sides of this equation

   $$\begin{aligned} \log((1 + g_{Y,t+1})Y_t) &= \log(Y_{t+1}) \\ \log(1 + g_{Y,t+1}) + \log(Y_t) &= \log(Y_{t+1}). \end{aligned}$$

   Hence, we get

   $$\log(1 + g_{Y,t+1}) = \log(Y_{t+1}) - \log(Y_t). \tag{1}$$

   Using the Taylor first-order approximation we can show that $\log(1 + g_{Y,t+1}) \approx g_{Y,t+1}$. For the logarithmic function when $x \to x_0$,

   $$\log(x) \approx \log(x_0) + \frac{1}{x_0}(x - x_0).$$

Now take a function argument $x = 1 + g_{Y,t+1}$ and approximate the logarithmic function around a point $g_{Y,t+1} = 0$ and thus have $x_0 = 1 + 0 = 0$. Then the approximation is

$$\log(1 + g_{Y,t+1}) \approx \log(1 + 0) + \frac{1}{1+0}(1 + g_{Y,t+1} - 1)$$
$$\log(1 + g_{Y,t+1}) \approx g_{Y,t+1}.$$

Plug the result in (1) to finish the proof

$$\log(1 + g_{Y,t+1}) \approx g_{Y,t+1} = \log(Y_{t+1}) - \log(Y_t).$$

This approximation result holds only for small values of the growth rate $g_{Y,t+1}$ (below 5%).

2. Suppose at period 0 GDP equals 100 and GDP grows at a constant rate of 3% per year. In how many years will GDP double? (Give an exact or an approximate answer.)

   Solution. Recall that GDP of period t can be found using GDP of period 0 and compound growth rate:

$$Y_t = Y_0(1 + g_Y)^t$$
$$\frac{Y_t}{Y_0} = (1 + g_Y)^t.$$

   Take logs on both sides of the equation and then express t:

$$\log\left(\frac{Y_t}{Y_0}\right) = \log\left((1 + g_Y)^t\right)$$
$$\log\left(\frac{Y_t}{Y_0}\right) = t\log(1 + g_Y)$$
$$t = \frac{\log\left(\frac{Y_t}{Y_0}\right)}{\log(1 + g_Y)}.$$

   The formula we have derived will provide an exact solution. When g_Y is small, we can use approximation $\log(1 + g_Y) \approx g_Y$. Then, the formula becomes

$$t \approx \frac{\log\left(\frac{Y_t}{Y_0}\right)}{g_Y}.$$

   The exact answer is

$$t = \frac{\log(2)}{\log(1 + 0.03)} = 23.45 \text{ years},$$

   while an answer when we use approximation is

$$t \approx \frac{\log(2)}{0.03} = 23.11 \text{ years}.$$

   The two answers are close to each other because in the example g_Y is small.

3. The growth rate of GDP, Y_t, equals 4%, and the growth rate of the population, N_t, equals 1%. Derive the formula for the growth rate of GDP per capita, $y_t = Y_t/N_t$, using the quotient (ratio) logarithm rule and the result from Exercise 1. Compute the growth rate

of GDP per capita. Note that the formula gives a good approximation only when the growth rates are small.

Solution. Using the log-difference approximation of the growth rate and the rules of logarithms, we can derive the growth rate of GDP per capita as a function of the GDP growth rate and the population growth rate:

$$
\begin{aligned}
&g_{y,t+1} \approx \log(y_{t+1}) - \log(y_t) \\
&g_{y,t+1} \approx \log\left(\frac{Y_{t+1}}{N_{t+1}}\right) - \log\left(\frac{Y_t}{N_t}\right) = \\
&= (\log(Y_{t+1}) - \log(N_{t+1})) - (\log(Y_t) - \log(N_t)) = \\
&= (\log(Y_{t+1}) - \log(Y_t)) - (\log(N_{t+1}) - \log(N_t)) = \\
&\approx g_{Y,t+1} - g_{N,t+1}.
\end{aligned}
$$

Note that the derived formula gives a good approximation when the growth rates of Y_t and N_t are small. Then, for a given example $g_{y,t+1} = 4\% - 1\% = 3\%$.

4. Suppose that production follows a Cobb-Douglas technology: $Y_t = B_t K_t^\alpha L_t^{1-\alpha}$, where B_t denotes productivity in period t, K_t - capital in period t, and L_t - labor in period t. Derive a growth rate of Y_t as a function of growth rates of B_t, K_t, and L_t. What is the contribution of labor growth to GDP growth?

   Solution. Using the log-difference approximation of the growth rate and the rules of logarithms, we can derive the growth rate of GDP as a function of the productivity growth rate, the capital growth rate, and the labor growth rate:

$$
\begin{aligned}
&g_{Y,t+1} \approx \log(Y_{t+1}) - \log(Y_t) = \log(B_{t+1} K_{t+1}^\alpha L_{t+1}^{1-\alpha}) - \log(B_t K_t^\alpha L_t^{1-\alpha}) = \\
&= \log(B_{t+1}) + \alpha \log(K_{t+1}) + (1-\alpha)\log(L_{t+1}) - (\log(B_t) + \alpha \log(K_t) + (1-\alpha)\log(L_t)) = \\
&\approx g_{B,t+1} + \alpha g_{K,t+1} + (1-\alpha) g_{L,t+1}.
\end{aligned}
$$

   The contribution of labor growth to GDP growth equals $(1-\alpha)g_{L,t+1}$.

5. Show that in continuous time taking a logarithm of a function $x(t)$ of time and then taking the derivative with respect to time t gives the growth rate. Apply this approach to derive the growth rate of GDP per capita as a function of the growth rate of GDP and the growth rate of the population. What are the benefits of working in continuous time?

   Solution. Let x_t at any point of time be denoted as $x(t)$, where t can take any value $t \in [0, +\infty)$. Then, the change of variable x per unit of time is given by

$$\frac{x(t+\Delta) - x(t)}{\Delta},$$

   where Δ is a time interval. If we drive the size of the time interval Δ to zero, we get the derivative of function $x(t)$ with respect to time t.

$$\dot{x}(t) = \lim_{\Delta \to 0} \frac{x(t+\Delta) - x(t)}{\Delta},$$

   where $\dot{x}(t)$ denotes $\frac{dx(t)}{dt}$. In continuous time the growth rate of a variable $x(t)$ is then given as

$$g_x(t) = \frac{\dot{x}(t)}{x(t)}.$$

As in discrete time, the growth rate is equal to the change in the variable divided by the level. Just that now the change is given by the time derivative.
Now let's prove that if we take a logarithm of $x(t)$ and then take a derivative with respect to time t, we will get the growth rate of a variable $x(t)$:

$$\frac{d\log(x(t))}{dt} = \frac{1}{x(t)}\frac{dx(t)}{dt} = \frac{\dot{x}(t)}{x(t)} = g_x(t).$$

Application for the growth rate of GDP per capita:

$$\begin{aligned}
&\log(y(t)) = \log(Y(t)) - \log(N(t)) \\
&\frac{d\log(y(t))}{dt} = \frac{d\log(Y(t))}{dt} - \frac{d\log(N(t))}{dt} \\
&\frac{\dot{y}(t)}{y(t)} = \frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{N}(t)}{N(t)} \\
&g_y(t) = g_Y(t) - g_N(t).
\end{aligned}$$

In continuous time, the obtained formula is exact (not an approximation).