ECON 702 Macroeconomics I

Discussion Handout 3*

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Content review

- Chapter 3 in the textbook FVK.
- Endogenous Labor Supply.
- Firm Problem.

Exercise 1. Endogenous Labor Supply

In this exercise, we consider a modified version of the household maximization problem. The new assumption is that households have a choice of how many hours to work, l_t. Now the agents will choose how much to consume, c_t, how much to save/borrow, a_{t+1}, and how much to work, l_t. The household maximization problem now reads as

$$\max_{\{c_t, a_{t+1}, l_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t, l_t)$$

subject to

$$\begin{aligned} & c_t + a_{t+1} = w_t l_t + (1 + r_t) a_t \quad \forall t = 0, 1, ..., T \\ & c_t \geq 0 \quad \forall t = 0, 1, ..., T \\ & a_{T+1} = 0 \\ & a_0 \text{ is given} \\ & l_t \geq 0 \quad \forall t = 0, 1, ..., T \end{aligned}$$

1. The new assumption on utility function is that $u(c_t, l_t)$ is strictly decreasing in l_t $\left(\frac{\partial u(c_t, l_t)}{\partial l_t} < 0\right)$. What trade-off does this assumption create? What are other basic assumptions on $u(c_t, l_t)$?

*Teaching Assistants: Anna Lukianova (Email:lukianova@wisc.edu) and John Ryan (Email: john.p.ryan@wisc.edu). Based on the lecture notes by Jesus Fernandez-Villaverde and Dirk Krueger.

Solution.

On the one hand, higher working hours allow households to get a higher labor income and, consequently, have more resources to spend on consumption and get higher utility from consumption (all else fixed). On the other hand, households get disutility from working. This creates a trade-off.

Other standard assumptions on $u(c_t, l_t)$ are: $u(c_t, l_t)$ is twice differentiable in c_t and in l_t(the first and the second derivatives exist), strictly increasing in c_t $\left(\frac{\partial u(c_t,l_t)}{\partial c_t} > 0\right)$, strictly decreasing in l_t $\left(\frac{\partial u(c_t,l_t)}{\partial l_t} < 0\right)$, strictly concave in c_t $\left(\frac{\partial^2 u(c_t,l_t)}{\partial c_t^2} < 0\right)$, strictly concave in l_t $\left(\frac{\partial^2 u(c_t,l_t)}{\partial l_t^2} < 0\right)$, and $u(c_t, l_t)$ satisfies the Inada conditions: $\lim_{c_t \to 0} \frac{\partial u(c_t,l_t)}{\partial c_t} = +\infty$, $\lim_{c_t \to \infty} \frac{\partial u(c_t,l_t)}{\partial c_t} = 0$ (similar for l_t).

2. Set up the Lagrangian and derive the intra-temporal (within-period) optimality condition. Provide an interpretation of the obtained optimality condition.

Solution.

The Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t, l_t) + \sum_{t=0}^{T} \lambda_t (w_t l_t + (1 + r_t) a_t - c_t - a_{t+1}).$$

For now, we ignore the condition $a_{T+1} = 0$, we can drop the constraints $c_t \geq 0$, $l_t \geq 0$ because the utility function satisfies the Inada conditions.

To derive the intra-temporal optimality condition we take the first-order conditions (FOC) with respect to c_t and l_t in any arbitrary period t:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u_c(c_t, l_t) - \lambda_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial l_t} = \beta^t u_l(c_t, l_t) + \lambda_t w_t = 0,$$

where $u_c(c_t, l_t)$ denotes the partial derivative of $u(c_t, l_t)$ with respect to c_t $\left(\frac{\partial u(c_t,l_t)}{\partial c_t}\right)$, and $u_l(c_t, l_t)$ stands for the partial derivative $u(c_t, l_t)$ with respect to l_t $\left(\frac{\partial u(c_t,l_t)}{\partial l_t}\right)$. Remember that $u_c(c_t, l_t) > 0$, and $u_l(c_t, l_t) < 0$. Combining these two conditions, we get the following intra-temporal (within-period) optimality condition:

$$\frac{u_l(c_t, l_t)}{u_c(c_t, l_t)} = -w_t. \tag{1}$$

Interpretation: the condition equates the marginal rate of substitution between two goods, consumption and labor, to its relative price. More intuitively, it is to think about leisure, the opposite of labor. One unit of leisure costs w_t units of consumption since this is the forgone wage from not working for one hour.

3. Find optimal labor supply using the following functional form of the utility function:

$$u(c_t, l_t) = \log\left(c_t - \psi \frac{l_t^{1+\eta}}{1+\eta}\right), \tag{2}$$

where the parameters $\psi \geq 0$ and $\eta \geq 0$ control the disutility of work. While ψ shapes the level of disutility, η controls how fast disutility increases with l_t. How does optimal labor supply depend on wage, w? Does labor supply depend on the parameters ψ and η?

Notes: the utility function (2) is an example of Greenwood, Hercowitz, and Huffman (GHH) preferences. The class of GHH preferences is broader: any preferences $u(c,l) = U(c - G(l))$ such that $U' > 0, U'' < 0, G' > 0, G'' > 0$. For this specific function, we can separate the optimal consumption (savings) choice and the optimal labor (leisure) choice. There is no wealth effect on labor supply.

Solution.

The partial derivatives of the GHH function (2) are

$$u_c(c_t, l_t) = \frac{1}{c_t - \psi \frac{l_t^{1+\eta}}{1+\eta}}$$
$$u_l(c_t, l_t) = \frac{1}{c_t - \psi \frac{l_t^{1+\eta}}{1+\eta}} \cdot \left(-\psi l_t^{\eta}\right).$$

Then the intra-temporal optimiality condition (1) becomes

$$-\psi l_t^{\eta} = -w_t. \tag{3}$$

Hence, optimal labor supply of a household is given by

$$l_t = \left(\frac{w_t}{\psi}\right)^{\frac{1}{\eta}}. \tag{4}$$

Optimal labor supply is increasing in wage $\left(\frac{\partial l_t}{\partial w_t} > 0\right)$, meaning that households work more when wage w_t is higher. The strength of the impact of wage on optimal labor supply is governed by $1/\eta$. So, higher η implies lower responsiveness of labor supply to wage changes. The optimal labor supply is decreasing in the disutility parameter ψ. Households with higher values of ψ get more disutility from work and choose to work less than households with lower values of ψ.

Note that optimal labor supply is independent of the optimal consumption choice independent of the optimal consumption choice (and thus independent of all other elements in the household budget constraint, including the wealth a_t of a household). That is, with this utility there is no so-called wealth effect on labor supply: with the same hourly wage w_t households with lots of wealth and those with little or no wealth find it optimal to work the same hours.

Remember that it is not always possible to solve for labor and consumption separately. The result is driven by the choice of preferences.

4. Given a solution for labor, derive the optimality condition for a consumption-saving decision. Provide an interpretation for the Euler equation (intertemporal, that is between-period, optimality condition).

Solution.

Given a solution for optimal labor supply, we can rewrite the budget constraint substituting l_t with (4) as follows

$$c_t + a_{t+1} = w_t^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} + (1 + r_t) a_t.$$

Setting up the Lagrangian as before now we have

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t, l_t) + \sum_{t=0}^{T} \lambda_t (w_t^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} + (1 + r_t)a_t - c_t - a_{t+1}).$$

To solve for optimal consumption c_t and assets $a_t + 1$ we need to find the inter-temporal optimality condition, also known as the Euler equation. The first-order conditions (FOC) with respect to c_t, c_{t+1}, and a_{t+1} in any arbitrary period t:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{c_t} &= \beta^t u_c(c_t, l_t) - \lambda_t = 0 \\
\frac{\partial \mathcal{L}}{c_{t+1}} &= \beta^{t+1} u_c(c_{t+1}, l_{t+1}) - \lambda_{t+1} = 0 \\
\frac{\partial \mathcal{L}}{a_{t+1}} &= \lambda_{t+1}(1 + r_{t+1}) - \lambda_t = 0.
\end{aligned}$$

Plugging in the expression for λ_t from the first condition and the expression for λ_{t+1} from the second equation into the third equation we receive the Euler equation:

$$u_c(c_t, l_t) = \beta(1 + r_{t+1}) u_c(c_{t+1}, l_{t+1}). \tag{5}$$

Interpretation of the Euler equation: if a household behaves optimally, it should be indifferent between consuming one unit today and saving it for consumption tomorrow.

For the GHH preferences, the Euler equation becomes

$$\frac{1}{c_t - \psi \frac{l_t^{1+\eta}}{1+\eta}} = \beta(1 + r_{t+1}) \frac{1}{c_{t+1} - \psi \frac{l_{t+1}^{1+\eta}}{1+\eta}}. \tag{6}$$

After plugging the optimal labor supply we have the following optimality condition

$$\beta(1 + r_{t+1}) \left(c_t - \frac{\psi}{1+\eta} \left(\frac{w_t}{\psi} \right)^{\frac{1+\eta}{\eta}} \right) = c_{t+1} - \frac{\psi}{1+\eta} \left(\frac{w_{t+1}}{\psi} \right)^{\frac{1+\eta}{\eta}}.$$

5. Characterize optimal consumption and assets under assumptions $(1 + r_t)\beta = 1$, $w_t = w_{t+1} = ... = w$, $r_t = r_{t+1} = ... = r$. Is the economy in a steady state?

Solution.

Under the assumptions $\beta(1 + r_t) = 1$, $r_t = r_{t+1} = ... = r$, and $w_t = w_{t+1} = ... = w$, we get

$$c_t - \frac{\psi}{1+\eta} \left(\frac{w}{\psi} \right)^{\frac{1+\eta}{\eta}} = c_{t+1} - \frac{\psi}{1+\eta} \left(\frac{w}{\psi} \right)^{\frac{1+\eta}{\eta}},$$

which implies

$$c_t = c_{t+1}.$$

The found solution describes a steady state since prices and allocation of consumption are constant across periods.

We can show that consumption c constitutes a constant share of the present discounted value of the lifetime income. This case illustrates the permanent income hypothesis (also known as Friedman's hypothesis). To find c and provide a complete characterization we need to proceed in the following order:

(a) Derive the intertemporal budget constraint, which says that the present discounted value of the lifetime consumption has to equal the present discounted value of the lifetime income and initial assets (wealth).

(b) Using the case of the steady state, we will obtain the formula for consumption c.

(c) When we know the optimal consumption c and the optimal labor supply, we can find assets for each period.

Below you can find solutions for each of the steps.

(a) Derivation of the intertemporal budget constraint.

Briefly, we are going to use the budget constraints: express assets in period $t+1$ from the budget constraint in period $t+1$ and plug this expression into the budget constraint in period t. We start with the last period budget constraint $t = T < \infty$. We know a_{T+1} has to equal 0. To keep the derivation more general we will have time subscripts for consumption, wage, and interest rate. Later we will utilize our case of a steady state and will pin down consumption. So, for $t = T$:

$$c_T + a_{T+1} = w_T^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} + (1 + r_T) a_T, \text{ where } a_{T+1} = 0 \rightarrow$$
$$c_T = w_T^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} + (1 + r_T) a_T$$
$$a_T = \frac{\left(c_T - w_T^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} \right)}{1 + r_T}.$$

If we plug it into the budget constraint of the period $t = T - 1$, we will be able to express a_{T-1}:

$$c_{T-1} + \underbrace{\frac{\left(c_T - w_T^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} \right)}{1 + r_T}}_{a_T} = w_{T-1}^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} + (1 + r_{T-1}) a_{T-1} \rightarrow$$
$$a_{T-1} = \frac{1}{1 + r_{T-1}} \left(c_{T-1} + \frac{\left(c_T - w_T^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} \right)}{1 + r_T} - w_{T-1}^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} \right)$$
$$a_{T-1} = \frac{1}{1 + r_{T-1}} \left(\left(c_{T-1} - w_{T-1}^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} \right) + \frac{\left(c_T - w_T^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} \right)}{1 + r_T} \right).$$

If we continue deriving assets for periods $t = T-2, T-3, T-4, ..., 1$ as a function of the gap between consumption and labor income, we can get the following formula for a_1:

$$a_1 = \frac{1}{1 + r_1} \left(c_1 - w_1^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} \right) + \frac{1}{(1 + r_1)(1 + r_2)} \left(c_2 - w_2^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} \right) + ... +$$
$$+ \frac{1}{(1 + r_1) \cdot ... \cdot (1 + r_T)} \left(c_T - w_T^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} \right) = \sum_{t=1}^{T} \frac{\left(c_t - w_t^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} \right)}{\Pi_{i=1}^{t} (1 + r_i)}.$$

Note: you can try to derive an intertemporal budget constraint for 3 periods to understand better the obtained formula for the case of T periods.

Then, using the budget constraint for $t = 0$, we express a_1 as follows

$$a_1 = \left(w_0^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} + (1 + r_0)a_0 - c_0 \right).$$

Combining last two formulas we have derived, we can show that the present value of lifetime consumption equals the present value of lifetime income and initial assets:

$$c_0 + \sum_{t=1}^{T} \frac{1}{\Pi_{i=1}^{t}(1 + r_i)} c_t = w_0^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} + \sum_{t=0}^{T} \frac{1}{\Pi_{i=1}^{t}(1 + r_i)} w_t^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} + (1 + r_0)a_0. \quad (7)$$

(b) Finally, if we go back to our special case with $w_t = w$, $r_t = r$, and $c_t = c$, we can get the simplified intertemporal budget constraint and show that consumption in each period constitutes a constant share of the present value of the lifetime income.

$$\sum_{t=0}^{T} \frac{1}{(1 + r)^t} c = \sum_{t=0}^{T} \frac{1}{(1 + r)^t} w^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} + (1 + r)a_0$$

$$c \sum_{t=0}^{T} \frac{1}{(1 + r)^t} = w^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} \sum_{t=0}^{T} \frac{1}{(1 + r)^t} + (1 + r)a_0$$

$$c = \frac{w^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} \sum_{t=0}^{T} \frac{1}{(1+r)^t} + (1 + r)a_0}{\sum_{t=0}^{T} \frac{1}{(1+r)^t}} = w^{1+\frac{1}{\eta}} \psi^{-\frac{1}{\eta}} + \frac{1 + r}{\sum_{t=0}^{T} \frac{1}{(1+r)^t}} a_0.$$

Consumption constitutes a constant fraction $\frac{1}{\sum_{t=0}^{T} \frac{1}{(1+r)^t}}$ of the present discounted value of the lifetime income and initial assets. Also, since labor income is identical in all periods, we can notice that in each period consumption equals labor income and a fraction of initial assets.

(c) Remember that a_0 is given, as well as w and r. When we know the optimal steady state consumption level we can find assets from the budget constraints. For example, the optimal choice of assets in period $t = 1$ is:

$$a_1 = (1 + r)a_0 - \frac{1 + r}{\sum_{t=0}^{T} \frac{1}{(1+r)^t}} a_0 = \frac{(1 + r)a_0 \left(\sum_{t=0}^{T} \frac{1}{(1+r)^t} - 1 \right)}{\sum_{t=0}^{T} \frac{1}{(1+r)^t}}.$$

Exercise 2. Firm Problem

1. For a neoclassical Cobb-Douglas production function

$$y_t = f(k_t, l_t) = k_t^\alpha (A_t l_t)^{1-\alpha}, \tag{8}$$

where $\alpha \in (0,1)$, y_t is real output, k_t is capital, l_t is labor, and A_t denotes the level of technology (TFP), show the following properties:

(a) the function exhibits constant returns to scale;

(b) both inputs are essential for the production technology;

(c) marginal product of labor and marginal product of capital are positive but decreasing;

(d) Inada conditions are satisfied.

Solution.

(a) For $\lambda > 1$, we need to check that $f(\lambda k_t, \lambda l_t) = \lambda f(k_t, l_t)$.

$$f(\lambda k_t, \lambda l_t) = (\lambda k_t)^\alpha (A_t(\lambda l_t))^{1-\alpha} = \lambda^a k_t^\alpha \lambda^{1-\alpha} (A_t l_t)^{1-\alpha} = \lambda k_t^\alpha (A_t l_t)^{1-\alpha} = \lambda f(k_t, l_t),$$

which implies that the function exhibits constant returns to scale.

(b) We need to show that $f(0, l_t) = 0$ and $f(k_t, 0) = 0$.

$$f(0, l_t) = 0^\alpha (A_t l_t)^{1-\alpha} = 0$$
$$f(k_t, 0) = k_t^\alpha (A_t \cdot 0)^{1-\alpha} = 0.$$

(c) We need to show that $\frac{\partial f(k_t, l_t)}{\partial k_t} > 0$, but $\frac{\partial^2 f(k_t, l_t)}{\partial k_t^2} < 0$, and similarly for labor.

$$\frac{\partial f(k_t, l_t)}{\partial k_t} = \alpha k_t^{\alpha-1} (A_t l_t)^{1-\alpha} > 0,$$
$$\frac{\partial^2 f(k_t, l_t)}{\partial k_t^2} = \alpha(\alpha - 1) k_t^{\alpha-2} (A_t l_t)^{1-\alpha} < 0,$$

since $\alpha \in (0,1)$ implying that $\alpha - 1 < 0$.
Similarly, for labor

$$\frac{\partial f(k_t, l_t)}{\partial l_t} = (1-\alpha) k_t^\alpha A_t (A_t l_t)^{-\alpha} > 0,$$
$$\frac{\partial^2 f(k_t, l_t)}{\partial l_t^2} = (1-\alpha)(-\alpha) k_t^\alpha A_t^2 (A_t l_t)^{-\alpha-1} < 0,$$

since $1 - \alpha > 0$ and $-\alpha < 0$.

(d) The Inada conditions are satisfied:

$$\lim_{k_t \to 0} \frac{\partial f(k_t, l_t)}{\partial k_t} = \lim_{k_t \to 0} \alpha k_t^{\alpha-1} (A_t l_t)^{1-\alpha} = \infty,$$
$$\lim_{k_t \to \infty} \frac{\partial f(k_t, l_t)}{\partial k_t} = \lim_{k_t \to \infty} \alpha k_t^{\alpha-1} (A_t l_t)^{1-\alpha} = 0,$$
$$\lim_{l_t \to 0} \frac{\partial f(k_t, l_t)}{\partial l_t} = \lim_{l_t \to 0} (1-\alpha) k_t^\alpha A_t (A_t l_t)^{-\alpha} = \infty,$$
$$\lim_{l_t \to \infty} \frac{\partial f(k_t, l_t)}{\partial l_t} = \lim_{l_t \to \infty} (1-\alpha) k_t^\alpha A_t (A_t l_t)^{-\alpha} = 0.$$

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2. What is an interpretation of α and $1-\alpha$?

   Solution.

   α is the elasticity of output with respect to the capital input, and $1-\alpha$ is the elasticity of output with respect to the labor input.

   In addition, in the competitive equilibrium, α equals the capital income share ($\mu_t k_t/y_t$), while $1-\alpha$ equals the labor income share ($w_t l_t/y_t$). How can we get this result? This is useful for getting a value for α from data.

3. Consider a representative firm, owned by households, that hires workers at wage w_t per unit of time, rents capital at rate μ_t, and produces the final good. Capital wears out in production at rate δ, implying that the return on capital for households is $r_t = \mu_t - \delta$. Solve the firm profit maximization problem:

$$\max_{l_t,k_t}\{y_t - w_t l_t - \mu_t k_t\}$$

subject to

$$y_t = k_t^{\alpha}(A_t l_t)^{1-\alpha}$$
$$k_t, l_t \geq 0 \quad \forall t = 0, 1, ..., T.$$

   Solution.

   The firm problem can be rewritten as follows

$$\max_{l_t,k_t}\left\{k_t^{\alpha}(A_t l_t)^{1-\alpha} - w_t l_t - \mu_t k_t\right\}.$$

   The FOCs with respect to capital and labor are

$$\alpha k_t^{\alpha-1}(A_t l_t)^{1-\alpha} = \mu_t$$
$$(1-\alpha)k_t^{\alpha}A_t^{1-\alpha}l_t^{-\alpha} = w_t.$$

   We cannot pin down the solutions for capital and labor separately, but we can find optimal capital-to-labor ratio by dividing the FOC with respect to labor by the FOC with respect to capital:

$$\frac{(1-\alpha)k_t^{\alpha}A_t^{1-\alpha}l_t^{-\alpha}}{\alpha k_t^{\alpha-1}(A_t l_t)^{1-\alpha}} = \frac{w_t}{\mu_t}$$
$$\frac{1-\alpha}{\alpha}\frac{k_t}{l_t} = \frac{w_t}{\mu_t}$$
$$\frac{k_t}{l_t} = \frac{\alpha}{1-\alpha}\frac{w_t}{\mu_t}.$$

   Note the right-hand side does not depend on any firm characteristics, which implies that all firms in equilibrium behave identically.

4. Provide an interpretation of the optimality conditions for labor and capital choices.

   Solution.

   In question 4. we have found the FOC with respect to labor:

$$w_t = \underbrace{(1-\alpha)k_t^{\alpha}A_t^{1-\alpha}l_t^{-\alpha}}_{f_l(k_t,l_t)=MPL}$$

   Marginal costs of using one additional unit of labor (that is wage w_t a firm has to pay) have equal marginal benefits of using one additional unit of labor (that is marginal product of labor $(1-\alpha)k_t^\alpha A_t^{1-\alpha} l_t^{-\alpha})$. Similarly, we can interpret the first-order condition for capital:

$$\mu_t = \underbrace{\alpha k_t^{\alpha-1}(A_t l_t)^{1-\alpha}}_{f_k(k_t,l_t)=MPK}.$$

5. Show that when a firm behaves optimally, its profits equal zero.
   Solution.
   In the equilibrium, we know what prices are, so the profits can be rewritten as

$$k_t^\alpha (A_t l_t)^{1-\alpha} - w_t l_t - \mu_t k_t = k_t^\alpha (A_t l_t)^{1-\alpha} - \underbrace{(1-\alpha)k_t^\alpha A_t^{1-\alpha} l_t^{-\alpha}}_{w_t} l_t - \underbrace{\alpha k_t^{\alpha-1} A_t^{1-\alpha} l_t^{1-\alpha}}_{\mu_t} k_t =$$

$$= k_t^\alpha (A_t l_t)^{1-\alpha} - (1-\alpha)(k_t^\alpha (A_t l_t)^{1-\alpha}) - \alpha k_t^\alpha (A_t l_t)^{1-\alpha} = 0.$$