ECON 702 Macroeconomics I

Discussion Handout 4 Solutions *

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1 Content review

- The **social planner's problem** is to maximize utility of the household subject to resource feasibility. (Assume $A_t = 1$ for notation)

$$\begin{aligned} &\max_{\{c_t, a_{t+1}\}} \sum_{t=0}^{T} \beta^t u(c_t) \\ \text{s.t.}\ & c_t + k_{t+1} = k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t, \quad \forall t \\ & c_t \geq 0, \quad \forall t \\ & 1 \geq l_t \geq 0, \quad \forall t \\ & k_{T+1} = 0 \\ & k_0 \text{ is given} \end{aligned}$$

- The **first welfare theorem** states that if a competitive equilibrium exists and there are no externalities, market power, or imperfect information, then the competitive equilibrium allocation solves the social planner's problem.
  - This is proved by showing that the Euler equation of the household in CE matches the optimality condition of the social planner.

$$1 + r_{t+1} = \alpha k_{t+1}^{\alpha-1} + 1 - \delta$$

- The **second welfare theorem** states that the allocation which solves the social planner's problem can be implemented as a competitive equilibrium.
  - This is proved by finding some prices $\{r_t, w_t\}$ such that the allocation as a function of prices matches the solution to the social planner's problem.
- The **Golden Rule** is the steady state level of capital that maximizes period consumption, $k^g = (\frac{\alpha}{\delta})^{\frac{1}{1-\alpha}}$. However, this does not account for household discounting. The **modified Golden Rule** is the steady state capital which maximizes lifetime utility of the household including discounting, $k^* = (\frac{\alpha}{\delta+\rho})^{\frac{1}{1-\alpha}}$.

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2 Perfect complements production

Consider an environment such as that which we studied in class with a representative household (exogenous labor supply $l = 1$, initial assets $a_0 > 0$), except the final good is produced with a Leontief production function, $y_t = f(k_t, l_t) = \min\{k_t, Al_t\}$.

1. Show that the production function demonstrates constant returns to scale and that both inputs are necessary for production.

   Solution: f has CRS iff $\forall \lambda > 0, f(\lambda x) = \lambda f(x)$.
   Let $\lambda > 0$.

   - Case 1: $k_t \leq Al_t$. So $f(t, Al_t) = \min\{k_t, Al_t\} = k_t$. Also, since $\lambda > 0$, we have $\lambda k_t < \lambda Al_t$. Thus, $f(\lambda k_t, \lambda Al_t) = \min\{\lambda k_t, \lambda Al_t\} = \lambda k_t = \lambda f(t, Al_t)$.
   - Case 2: $k_t > Al_t$. So $f(t, Al_t) = \min\{k_t, Al_t\} = Al_t$. Also, since $\lambda > 0$, we have $\lambda k_t > \lambda Al_t$. Thus, $f(\lambda k_t, \lambda Al_t) = \min\{\lambda k_t, \lambda Al_t\} = \lambda Al_t = \lambda f(t, Al_t)$.

   In both cases, we have the desired result, and thus f has constant returns to scale.

2. Define a competitive equilibrium.

   Solution:

   A competitive equilibrium given a_0 in this environment is an allocation for households $\{c_t, l_t, a_{t+1}\}_{\forall t}$, an allocation for firms $\{k_t, l_t\}_{\forall t}$, and prices $\{\mu_t, r_t, w_t\}_{\forall t}$, such that:

   (a) Given prices and initial assets, the household's allocation solves their utility maximization problem (standard, but should write for practice).

   (b) Given prices, the firm's allocation solves their profit maximization problem.

   $$\max_{k_t, l_t} \quad min\{k_t, Al_t\} - \mu_t k_t - w_t l_t$$

   (c) Markets clear $\forall t$:

   $$c_t + k_{t+1} = min\{k_t, Al_t\} + (1 - \delta)k_t$$
   $$k_t = a_t$$
   $$l_t = 1$$

3. Assume $\mu_t < 1$. What must be the capital choice of the firm in a competitive equilibrium? (Hint: the labor market must clear in equilibrium).

   Solution:

A competitive equilibrium requires that a firm chooses $l_t = 1$ since markets clear. Given that the firm is choosing this, the firm's marginal product of capital is $MPK = 1$ if $k_t < A$, and $MPK = 0$ otherwise. Thus, the firm will hire capital until marginal cost equals marginal revenue. Since $\mu_t < 1$ (by assumption, but can be verified later), the MC of hiring an addition unit of capital is lower than the MR until $k_t = A$. Thus, the firm's choice of capital in a competitive equilibrium in which $\mu_t < 1$ is $k_t = A$. Note that we can't use the typical $\mu_t = MPK$ since the marginal product of capital does not exist at $k_t = Al_t$.

4. In a steady state equilibrium with $\delta = 0$, and firms earning zero profit, what are the prices in the economy? (if it helps, you can assume that $\beta \in (.5, 1)$.

Solution:
Since $\delta = 0$, we have that $\mu_t = r_t$. Recall the household's Euler equation:

$$u'(c_t) = \beta(1 + r_{t+1})u'(c_{t+1})$$

In a steady state, we have that $c_t = c_{t+1}$, so we should have that

$$\beta(1 + r_{t+1}) = 1 \implies r_t = \frac{1}{\beta} - 1$$

Since the firm's choice of capital is $k_t = A$, and $\mu_t = r_t = \frac{1}{\beta} - 1$, the firm's profit is

$$min\{k_t, Al_t\} - \mu_t k_t - w_t l_t = A - (\frac{1}{\beta} - 1)A - w_t$$

. Since firm profits are zero, we have that $w_t = (2 - \frac{1}{\beta})$.

5. What are the allocations for the household and the firm in a steady state competitive equilibrium?

Solution: Using market clearing, we have that $a_t = a_{t+1} = A$. The households budget constraint is

$$c_t + a_{t+1} = w_t + (1 + r_t)a_t$$

Which gives us

$$c_t + A = (2 - \frac{1}{\beta})A + \frac{1}{\beta}A \implies c_t = A$$

So the allocations are $\{c_t, a_{t+1}\} = \{A, A\}$ for the household, and $\{k_t, l_t\} = \{A, 1\}$

6. Write the planner's problem.

Solution:

$$
\begin{aligned}
&\max_{\{c_t, k_{t+1}\}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{s.t. } & c_t + k_{t+1} = min\{k_t, Al_t\} + (1-\delta)k_t, \quad \forall t \\
& c_t \geq 0, \quad \forall t \\
& 1 \geq l_t \geq 0, \quad \forall t \\
& k_0 \text{ is given}
\end{aligned}
$$

7. Does the competitive equilibrium allocation solve the social planner's problem?

Solution:

This problem is not easy to solve for an arbitrary $k_0 < A$ since we cannot use first order conditions for $k_{\{t+1\}}$. However, since there is no market power, imperfect information, or externalities, we know that the competitive equilibrium allocation does solve the planner's problem by the first welfare theorem.