ECON 702 Macroeconomics I

Discussion Handout 6 *

Solution

8 March 2024

Balanced Growth Path

Suppose we have an economy in which households have CRRA utility, $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, the production technology is Cobb-Douglas $F(A, k, \ell) = k^{\alpha}(A\ell)^{1-\alpha}$, with exogenous growth in total factor productivity at rate g and full depreciation $\delta = 1$. There is exogenous labor supply (normalized to $\ell = 1$), and no population growth.

1. What does it mean for a variable to be growing at a constant rate? Write A_t as a function of g, t, and A_0. What about if we were in continuous time?

   Solution:
   In discrete time, A_t grows at a constant rate if and only if $A_t = (1+g)^t A_0$. In continuous time, a constant growth rate means that $\frac{\dot{A}(t)}{A(t)} = g$, which implies $A(t) = A_0 e^{gt}$. Sketch a graph of an exponential function here.

2. Are the household's Euler equations, the firm's optimality conditions, or market clearing conditions any different in this environment, compared to the model we studied before? Write these conditions.
   Solution:
   No, they aren't any different. Here are the household and firm optimality conditions:

$$
\begin{aligned}
u'(c_t) &= \beta(1+r_{t+1})u'(c_{t+1}) \\
\mu_t &= r_t + 1 \\
\mu_t &= F_k(A_t, k_t, 1) = \alpha \left(\frac{A_t}{k_t}\right)^{1-\alpha} \\
w_t &= F_\ell(A_t, k_t, 1) = (1-\alpha)k^{\alpha} A_0^{1-\alpha}
\end{aligned}
$$

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Here is the goods market clearing condition:

$$c_t + k_{t+1} = k_t^\alpha A_t^{1-\alpha}$$

We already used the labor market clearing condition in the goods market clearing and firm's first order conditions, and so by Walras law, the capital market clears as well.

3. Define a balanced growth path.
Solution:
A balanced growth path is a competitive equilibrium in which allocations and prices all grow at a constant rate:

$$\begin{aligned} c_t &= (1+g_c)^t c \\ k_t &= (1+g_k)^t k \\ y_t &= (1+g_y)^t k \\ r_t &= (1+g_r)^t r \\ w_t &= (1+g_w)^t w \end{aligned}$$

4. Under the conjecture that capital and consumption grow at the same rate as TFP, characterize a BGP in this economy. What is the growth rate of output?

Solution:
We conjecture that $c_t = (1+g)^t c$ and $k_t = (1+g)^t k$. Plugging in c_t and c_{t+1}, together with $u'(c) = c^{-\sigma}$, we get

$$\begin{aligned} ((1+g)^t)^{-\sigma} &= \beta(1+r_{t+1})((1+g)^{t+1})^{-\sigma} \\ \implies \beta(1+r_{t+1}) &= (1+g)^\sigma \end{aligned}$$

Plugging in $k_t = (1+g)^t k$ and $A_t = (1+g)^t A_0$ into the firm's first order condition for capital, we get that $1 + r_t = \alpha(\frac{A_0}{k})^{1-\alpha}$. We can plug this into the Euler equation to get

$$\begin{aligned} \beta\alpha\left(\frac{A_0}{k}\right)^{1-\alpha} &= (1+g)^\sigma \\ \implies k &= [\alpha\beta(1+g)^{-\sigma}]^{\frac{1}{1-\alpha}} A_0 \end{aligned}$$

Plugging in c_t, k_t, A_t into the goods market clearing, we get

$$\begin{aligned} c + (1+g)k &= k^\alpha A_0^{1-\alpha} \\ \implies c &= k^\alpha A_0^{1-\alpha} - (1+g)k \end{aligned}$$

We can also see that

$$\begin{aligned} y_t &= k_t^\alpha A_t^{1-\alpha} \\ &= (1+g)^{\alpha t} k^\alpha (1+g)^{(1-\alpha)t} A_0^{1-\alpha} \\ &= (1+g)^t y \end{aligned}$$

where $y = k^\alpha A_0^{1-\alpha}$. So y_t also grows at rate g.

5. Japan is a country with approximately zero population growth over the past 25 years, but their (real) GDP has grown from about 475 trillion yen to about 540 trillion yen from 1997 to 2022. What is the average annual growth rate in Japan over this time period? [1]

   Solution:
   $y_t = (1+g)^t y_0$, so $g = \left(\frac{y_t}{y_0}\right)^{1/t} - 1$, so $g_{Japan} \approx \frac{540}{475}^{1/25} - 1 \approx .005$, so Japan's growth rate has been about .5% over the past 25 years.

6. Data shows that in Japan, $\alpha \approx .43$, and real consumption in 1997 was about 258 trillion yen. [2] Using this information, give an estimate of k_{1997} and A_{1997} using our neoclassical model.

   Solution:
   Using 1997 as our base year, from the time 0 resource constraint: $c_{1997} = y_{1997} - (1+g)k_{1997}$. Thus, $k_{1997} = (y-c)/(1+g) = (475-258)/1.005 \approx 215.9$.
   From the production function, $y = k^\alpha A_0^{1-\alpha}$, so $A_{1997} = (y/k^\alpha)^{\frac{1}{1-\alpha}} \approx (475/215.9^{.43})^{1/.57} = 861.05$.

Transitional Dynamics

Now suppose that we do not start on a BGP.

1. Assume here that $\sigma = 1$, so $u(c) = \log c$. Let us denote our detrended variables $\hat{c}_t$, and $\hat{k}_t$ such that $c_t = (1+g)^t \hat{c}_t$ and $k_t = (1+g)^t \hat{k}_t$. Find the detrended Euler equation and resource constraint (in terms of $\hat{c}_t$ and $\hat{k}_t$).

   Solution:
   Same as slides, just plug $\hat{c}$ and $\hat{k}$ into the Euler equation and the resource constraint, get

$$\frac{\hat{c}_{t+1}}{\hat{c}_t} = \frac{\beta\alpha}{1+g}\left(\frac{A_0}{\hat{k}_{t+1}}\right)^{1-\alpha}$$
$$\hat{c}_t + (1+g)\hat{k}_{t+1} = \hat{k}_t^\alpha A_0^{1-\alpha}$$

2. Conjecture that $\hat{k}_{t+1} = s\hat{k}_t^\alpha$, and solve for s.

   Solution:
   Plug in $\hat{k}_{t+1}$ into the resource constraint and solve for $\hat{c}$, to get

$$\begin{aligned}\hat{c}_t &= \hat{k}_t^\alpha A_0^{1-\alpha} - (1+g)\hat{k}_{t+1} \\ &= \hat{k}_t^\alpha (A_0^{1-\alpha} - s(1+g))\end{aligned}$$

   So we have that

$$\begin{aligned}\frac{\hat{c}_{t_1}}{\hat{c}_t} &= \frac{\hat{k}_{t+1}^\alpha}{\hat{k}_t^\alpha} \\ &= \frac{\beta\alpha}{1+g}\left(\frac{A_0}{\hat{k}_{t+1}}\right)^{1-\alpha} \\ \implies \hat{k}_{t+1} &= \frac{\beta\alpha}{1+g} A_0^{1-\alpha}\hat{k}_t^\alpha\end{aligned}$$

[1] From FRED data.
[2] See FRED consumption data and labor income share statistics.

So we conclude that $s = \frac{\beta\alpha}{1+g} A_0^{1-\alpha}$.

3. Suppose $\beta = .9$ and our other primitives are those we solved for in Japan. Solve for the policy function of capital and consumption and sketch a graph. Find the fixed point of the policy function, where the detrended variables reach a steady state.

   Solution:
   $s = \frac{\beta\alpha}{1+g} A_0^{1-\alpha} = \frac{.9*.43}{1+.005} 861^{.57} \approx 18.1.$

   So the policy function is $\hat{k}_{t+1} = 18.1 \hat{k}_t^{.43}$.

   The fixed point is then where $\hat{k} = 18.1\hat{k}^{.43} \implies \hat{k}^{.57} = 18.1 \implies \hat{k} \approx 160.9$.

   Sketch a graph here of a concave function, along with $y = x$ to show the fixed point.