# ECON 702 Macroeconomics I

Discussion Handout 7 *

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Questions marked with asterisks (*) are additional questions.

## 1 The Solow Model

In the neoclassical growth model considered in the class so far, households (or the social planner for them) decide optimally how much to save. In other words, the saving rate $s$, which is the fraction of income (output) being saved, was chosen optimally. In the Solow model, the saving rate $s$ is a fixed number that is taken as given.

Consider an infinite-period environment, in which the households' preferences are described by the utility function $u(c)$, while final goods are produced with the Cobb-Douglas production technology: $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$. The population grows at the constant growth rate $n$ and TFP at the constant rate $g$. Households save (= invest) a fraction $s$ of output. Capital depreciates each period at the rate $\delta$.

1. Set up the social planner problem of the Solow model in per capita terms.

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2. Solve the model (find expressions for $c_t$, $i_t$, $k_{t+1}$, and $y_t$).

3. (*) Show that on the BGP, capital per capita and consumption per capita grow at the growth rate of TFP, $g$. Additionally, demonstrate that aggregate capital and aggregate consumption grow at the growth rate of $(g+n)$ (if $g$ and $n$ are small).

4. Derive the BGP initial capital stock per capita, consumption per capita, and output per capita assuming (and knowing from the previous question) that these variables grow at the constant growth rate of TFP, $g$. Hint: use the resource constraint and recall that the saving rate, $s$, determines how much to save and consume.

5. Find the BGP level of capital per capita that allows households to achieve the highest level of consumption in the long run. In other words, derive the golden rule capital stock.

6. Find the saving rate associated with the golden rule per capita capital stock, $s_{GR}$. Hint: recall that $s \equiv \frac{i}{y}$.

7. (*) What level of the saving rate in the Solow model guarantees that $k^{Solow}$ equals the BGP initial level of capital per capita, $k$, in the neoclassical growth model with CRRA preferences $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$?

8. (*) Compare the saving rate that maximizes consumption (and output) per capita, $s^{SR}$, with the optimal saving rate, $s$, that maximizes the lifetime utility of households with CRRA preferences $\left(u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}\right)$ in the neoclassical growth model. Interpret the results.

## 2 The AR(1) Process[1]

To study the short-run movements of the economy we introduce random technology shocks (also known as productivity shocks) in the neoclassical growth model. Assume that technology evolves as follows:

$$A_t = (1+g)^t e^{z_t} A_0,$$

where $g$ is a constant growth rate of technology and $z_t$ follows an autoregressive process of order 1 (abbreviated AR(1)) of the form:

$$z_t = \rho z_{t-1} + \sigma_\varepsilon \varepsilon_t,$$

where $0 < \rho < 1$, $\varepsilon_t$ is a productivity shock that comes from a standard normal distribution $\mathcal{N}(0,1)$, and for simplicity we assume initial conditions $A_0 = 1$ and $z_{-1} = 0$. $(1+g)^t$ that captures the long-run evolution of $A_t$ with a transitory component $e^{z_t}$.

1. Interpret $e^{z_t}$, $z_t$, $\varepsilon_t$, $\rho$, $\sigma$ in the context of the business cycle model.

2. What are an unconditional and a conditional expectation of $z_t$? Show that $\mathbf{E}z_t = 0$ and derive $\mathbf{E}_{t-1}z_t = \rho z_{t-1}$ for all $t$.

3. (*) Numerical exercise. Plot the path of the process for technology $z_t$ for 300 periods[2]. Draw 300 productivity shocks $\varepsilon$ from the normal distribution $N(0,1)$ and keep them the same when you plot the evolution of $z_t$ given the following values of the parameters $\rho$ and $\sigma_\varepsilon$:
   - (a) Low volatility of productivity shocks and low persistence: $\rho = 0.1$, $\sigma_\varepsilon = 0.002$
   - (b) High volatility of productivity shocks and low persistence: $\rho = 0.1$, $\sigma_\varepsilon = 0.008$
   - (c) Low volatility of productivity shocks and high persistence: $\rho = 0.96$, $\sigma_\varepsilon = 0.002$
   - (d) High volatility of productivity shocks and high persistence: $\rho = 0.96$, $\sigma_\varepsilon = 0.008$.

4. How do we measure productivity shocks under the assumptions of this model?

[1] You can learn more about autoregressive processes of higher order and autoregressive moving average models (ARMA) in the textbook (p. 213-214, Remark 63).

[2] Use can use Excel or another more sophisticated software to plot graphs (e.g., Julia, Python, Matlab, Stata).