ECON 702 Macroeconomics I

Discussion Handout 7 *

Solution

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The Solow Model

In the neoclassical growth model considered in the class so far, households (or the social planner for them) decide optimally how much to save. In other words, the saving rate s, that is the fraction of income (output) that is being saved, was chosen optimally. In the Solow model, the saving rate s is a fixed number that is taken as given.

Consider an infinite-period environment, in which the households' preferences are described by the utility function $u(c)$, while final goods are produced with the Cobb-Douglas production technology: $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$. The population grows at the constant growth rate n and TFP grows at the constant rate g. Households save a fraction s of output. Capital depreciates each period at rate δ.

1. Set up the social planner problem of the Solow model in per capita terms.

   Solution.

   To write the problem in per capita terms we need to formulate the aggregate output production function and the aggregate resource constraint in per capita terms. The production function in per capita terms:

$$y_t \equiv \frac{Y_t}{N_t} = \frac{K_t^{\alpha} A_t^{1-\alpha}}{N_t^{\alpha} N_t^{1-\alpha}} = \frac{K_t^{\alpha}}{N_t^{\alpha}} \left(A_t \frac{L_t}{N_t} \right)^{1-\alpha} = k_t^{\alpha} (A_t l_t)^{1-\alpha}$$

   The resource constraint in per capita terms:

$$\frac{C_t}{N_t} + \frac{K_{t+1}}{N_{t+1}} \underbrace{\frac{N_{t+1}}{N_t}}_{1+n} = \frac{K_t^{\alpha}(A_t L_t)^{1-\alpha}}{N_t} + (1-\delta)\frac{K_t}{N_t}$$

$$c_t + k_{t+1}(1+n) = k_t^{\alpha}(A_t l_t)^{1-\alpha} + (1-\delta)k_t.$$

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Note that in the model with population growth ($n \neq 0$) we have the term $(1+n)$ in front of capital per capita tomorrow. This is because to keep capital per capita k_{t+1} at the same level as k_t since there are n more people in period $t+1$ than in period t, society has to spend an extra nk_{t+1} units of resources.
Since the saving rule is exogenously given, the investments are known

$$I_t \equiv K_{t+1} - (1-\delta)K_t = sY_t.$$

In per capita terms:

$$i_t = (1+n)k_{t+1} - (1-\delta)k_t = sy_t.$$

The problem of the social planner who maximizes per capita lifetime utility subject to the resource constraint, production technology, and the saving rule is

$$\begin{aligned}
&\max_{c_t, l_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
&c_t + i_t = y_t \\
&y_t = k_t^{\alpha}(A_t l_t)^{1-\alpha} \\
&i_t = (1+n)k_{t+1} - (1-\delta)k_t \\
&i_t = sy_t \\
&A_0, k_0 \equiv \frac{K_0}{N_0} \text{ is given} \\
&c_t \geq 0, \quad 0 \leq l_t \leq 1, \quad k_{t+1} \geq 0
\end{aligned}$$

2. Solve the model (find expressions for c_t, i_t, k_{t+1}, y_t).

Solution.

First, notice that since the households do not value leisure in their utility function it is optimal for the social planner to let households work full-time. So, $l_t = 1$ for all $t = 0, 1, 2, ...$

Second, in the Solow model all endogenous variables are pinned down from the constraints of the social planner. Recall that we k_0 is given. Then,

$$y_0 = k_0^{\alpha} A_0^{1-\alpha}$$

is known. Hence, we can find investments as the saving rule is exogenously given:

$$i_0 = sy_0 = sk_0^{\alpha} A_0^{1-\alpha}.$$

Consumption is pinned down by the resource constraint:

$$c_0 = y_0 - i_0 = y_0 - sy_0 = (1-s)y_0.$$

Once we know investments i_0 we can find capital per capita in period 1:

$$k_1 = \frac{sy_0 + (1-\delta)k_0}{1+n}.$$

Then we can compute $A_1 = (1+g)A_0$ and find y_1, c_1, i_1 and so on. The policy function for capital is

$$k_{t+1} = \frac{sk_t^{\alpha} A_t^{1-\alpha} + (1-\delta)k_t}{1+n}$$

3. (*) Show that on the BGP capital per capita and consumption per capita grow at the growth rate of TFP, g. Show that aggregate capital and aggregate consumption grow at the growth rate of $g+n$ (if g and n are small).

Solution.

Let's assume that on the BGP consumption per capita grows at the constant rate g_c, and capital per capita grows at the constant rate g_k. Then, the resource constraint on the BGP is

$$(1+g_c)^t c + (1+n)(1+g_k)^{t+1}k = ((1+g_k)^t k)^\alpha ((1+g)^t A_0)^{1-\alpha} + (1-\delta)(1+g_k)^t k.$$

Divide the both sides of the equation by $(1+g_k)^t$:

$$\left(\frac{1+g_c}{1+g_k}\right)^t c + (1+n)(1+g_k)k = (1+g_k)^{t\alpha - t} k^\alpha ((1+g)^t A_0)^{1-\alpha} + (1-\delta)k$$

$$\left(\frac{1+g_c}{1+g_k}\right)^t c + [(1+n)(1+g_k) - (1-\delta)]k = \left[\left(\frac{1+g}{1+g_k}\right)^{1-\alpha}\right]^t k^\alpha.$$

This equation has to hold for any period. It is possible if and only if

$$\frac{1+g_c}{1+g_k} = \left(\frac{1+g}{1+g_k}\right)^{1-\alpha} = 1.$$

Otherwise, the left-hand side and the right-hand side will grow at different rates. Since $\alpha \in (0,1)$ the condition is true if $g_c = g_k = g$.

The growth rates of aggregate capital and aggregate consumption are

$$g_K = g_{k*N} \approx g_k + g_N = g + n$$
$$g_C = g_{c*N} \approx g_c + g_N = g + n.$$

The approximation is good if g and n are small.

4. Derive the BGP initial capital stock per capita, consumption per capita, and output per capita assuming (knowing from the previous question) that these variables grow at the constant growth rate of TFP, g. Hint: use the resource constraint and recall that the saving rate, s, dictates how much to save and consume.

Solution.

We assume (and know from the previous question) that on the BGP consumption per capita and capital per capita grow at rate g. That is, $c_t = (1+g)^t c, k_t = (1+g)^t k$ where c and k are some initial levels at the BGP. Hence, the resource constraint along the BGP can be written as follows:

$$(1+g)^t c + (1+n)(1+g)^{t+1}k = ((1+g)^t k)^\alpha ((1+g)^t A_0)^{1-\alpha} + (1-\delta)(1+g)^t k$$
$$c + (1+n)(1+g)k = k^\alpha A_0^{1-\alpha} + (1-\delta)k$$
$$c + [(1+n)(1+g) - (1-\delta)]k = A_0^{1-\alpha} k^\alpha.$$

In this equation, there are two unknowns: c and k. Now recall that in the Solow model the saving rate, s, is given and it dictates that $k = sy = sk^\alpha A_0^{1-\alpha}$, and $c = (1-s)y = (1-s)k^\alpha A_0^{1-\alpha}$. Plug c in the resource constraint to obtain level of capital, k.

$$(1-s)A_0^{1-\alpha}k^\alpha = A_0^{1-\alpha}k^\alpha - [(1+n)(1+g) - (1-\delta)]k$$
$$sA_0^{1-\alpha}k^\alpha = [(1+n)(1+g) - (1-\delta)]k$$
$$k = \left(\frac{[(1+n)(1+g) - (1-\delta)]}{sA_0^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}.$$

So, the BGP initial capital stock, consumption per capita, and output per capita in the Solow model are given by

$$k^{Solow} = \left(\frac{s^{Solow}A_0^{1-\alpha}}{[(1+n)(1+g)-(1-\delta)]}\right)^{\frac{1}{1-\alpha}}$$
$$c^{Solow} = \left(1-s^{Solow}\right)A_0^{1-\alpha}\left(k^{Solow}\right)^{\alpha} = (1-s^{Solow})A_0^{1-\alpha}\left(\frac{s^{Solow}A_0^{1-\alpha}}{[(1+n)(1+g)-(1-\delta)]}\right)^{\frac{\alpha}{1-\alpha}}$$
$$y^{Solow} = A_0^{1-\alpha}(k^{Solow})^{\alpha} = A_0^{1-\alpha}\left(\frac{s^{Solow}A_0^{1-\alpha}}{[(1+n)(1+g)-(1-\delta)]}\right)^{\frac{\alpha}{1-\alpha}}.$$

Note that the BGP capital per capita is increasing in the saving rate s^{Solow}. Why?

5. Find the BGP level of capital per capita that allows households to achieve the highest level of consumption in the long run. In other words, derive the golden rule capital stock.

Solution.

Recall that the BGP level of consumption in period t is $(1+g)^t c$. To maximize the long-run BGP level of consumption, we need to maximize the BGP initial level of consumption, c. From the resource constraint in the BGP, BGP consumption is given by

$$c = A_0^{1-\alpha}k^{\alpha} - [(1+n)(1+g)-(1-\delta)]k.$$

To determine the golden rule capital stock we take the FOC with respect to k on the right-hand side:

$$\alpha A_0^{1-\alpha}k^{\alpha-1} - [(1+n)(1+g)-(1-\delta)] = 0.$$

Hence,

$$k_{GR} = \left(\frac{\alpha A_0^{1-\alpha}}{(1+n)(1+g)-(1-\delta)}\right)^{\frac{1}{1-\alpha}}.$$

6. Find the saving rate associated with the golden rule per capita capital stock, s_{GR}. Hint: recall that $s \equiv \frac{i}{y}$.

Solution.

$$s_{GR} = \frac{i_{GR}}{y_{GR}} = \frac{((1+g)(1+n)-(1+\delta))k_{GR}}{k_{GR}^{\alpha}} = ((1+g)(1+n)-(1+\delta))k_{GR}^{1-\alpha}.$$

After plugging the derived formula for k_{GR}, we get

$$s_{GR} = \alpha.$$

Note that this result can be easily seen if we compare the formulas for k^{Solow} and k_{GR}.

7. (*) What level of the saving rate in the Solow model guarantees that k^{Solow} equals the BGP initial level of capital per capita, k, in the neoclassical growth model with CRRA preferences $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$?

Solution.

The BGP initial level of capital per capita in the neoclassical model with CRRA preferences $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ is

$$k = \left(\frac{\alpha\beta A_0^{1-\alpha}}{(1+n)(1+g)^\sigma - \beta(1-\delta)}\right)^{\frac{1}{1-\alpha}} = \left(\frac{\alpha A_0^{1-\alpha}}{\frac{1}{\beta}(1+n)(1+g)^\sigma - (1-\delta)}\right)^{\frac{1}{1-\alpha}}.$$

Note: recall how we derive k.
To answer the question we need to find s^{Solow} such that $k^{Solow} = k$:

$$\left(\frac{s^{Solow}A_0^{1-\alpha}}{[(1+n)(1+g) - (1-\delta)]}\right)^{\frac{1}{1-\alpha}} = \left(\frac{\alpha A_0^{1-\alpha}}{\frac{1}{\beta}(1+n)(1+g)^\sigma - (1-\delta)}\right)^{\frac{1}{1-\alpha}}.$$

After some math manipulation, we get the following:

$$\frac{s^{Solow}}{(1+n)(1+g) - (1-\delta)} = \frac{\alpha}{\frac{1}{\beta}(1+n)(1+g)^\sigma - (1-\delta)}$$
$$s^{Solow} = \frac{\alpha[(1+n)(1+g) - (1-\delta)]}{\frac{1}{\beta}(1+n)(1+g)^\sigma - (1-\delta)}.$$

8. (*) Compare the saving rate that maximizes consumption (and output) per capita, s_{SR}, with the optimal saving rate that maximizes the lifetime utility of households with CRRA preferences $\left(u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}\right)$, s. Interpret the results.

Solution. First, let's recall that the saving rate is the fraction of savings (= investments) in the output:

$$s = \frac{i}{y} = \frac{((1+g)(1+n) - (1-\delta))k}{k^\alpha A_0^{1-\alpha}} = ((1+g)(1+n) - (1-\delta))k^{1-\alpha}A_0^{-(1-\alpha)}.$$

Hence, to compare the saving rates s_{SR} with s that maximizes the lifetime utility in the neoclassical growth model we need to compare k_{GR} and k:

$$k = \left(\frac{\alpha A_0^{1-\alpha}}{\frac{1}{\beta}(1+n)(1+g)^\sigma - (1-\delta)}\right)^{\frac{1}{1-\alpha}}$$
$$k_{GR} = \left(\frac{\alpha A_0^{1-\alpha}}{(1+n)(1+g) - (1-\delta)}\right)^{\frac{1}{1-\alpha}}$$

$k_{GR} > k$ as long as

$$\frac{(1+g)^\sigma}{\beta} > (1+g).$$

This condition should hold for the existence of the social planner solution (why?). Then this implies that $s_{RG} > s$. If the social planner can choose the saving rate optimally (in the neoclassical growth model), then the saving rate for maximizing the lifetime utility is smaller than one for optimizing long-run consumption.
This happens because the social planner chooses a smaller initial level of capital on the BGP and if the economy starts with capital stock $k_0 < k$, less capital accumulation is required. Less capital accumulation means less investment and more consumption in the short run, which is beneficial for households, especially if they are impatient. Thus the consumption and utility losses in the BGP from having less consumption than the golden rule is (more than) compensated by the higher utility along the pathtowards the BGP (the so-called transition path).

The AR(1) Process[1]

Assume that technology evolves as follows:

$$A_t = (1+g)^t e^{z_t} A_0,$$

where g is a constant growth rate of technology and z_t follows as autoregressive process of order 1 (abbreviated AR(1)) of the form:

$$z_t = \rho z_{t-1} + \sigma_\varepsilon \varepsilon_t,$$

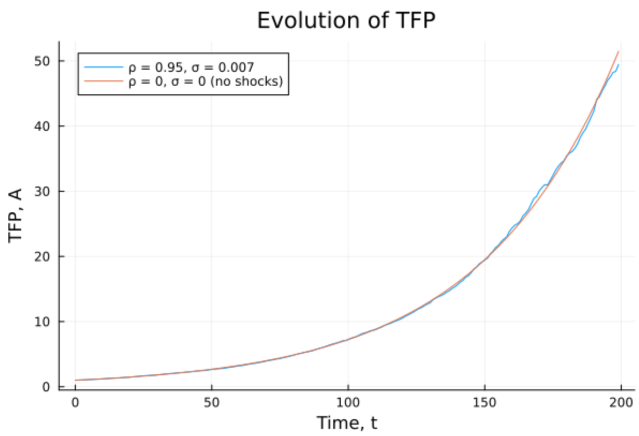
where $0 < \rho < 1$, ε_t is a productivity shock that comes from a standard normal distribution $\mathcal{N}(0,1)$, and for simplicity we assume initial conditions $A_0 = 1$ and $z_{-1} = 0$. $(1+g)^t$ that captures the long-run evolution of A_t with a transitory component e^{z_t}.

1. Interpret e^{z_t}, ε_t, ρ, σ.

   Solution.

   e^{z_t} is a component that governs the transitory dynamics of technology. When $e^{z_t} > 1$ (which happens when $z_t > 0$), the technology improves faster than average, and when $e^{z_t} < 1$ (which happens when $z_t < 0$), the technology improves less than average. If e^{z_t} is very small, that is the economy suffers a large negative shock, it may happen that $A_t < A_{t-1}$. z_t behaves randomly (there is a productivity shock every period, ε_t) but persistently (there is dependence between z_t and z_{t-1}). σ_ε is a parameter that controls the volatility of the shock ε_t: a large σ_ε means high volatility and a small σ_ε a low volatility. This affects deviations of z_t from zero. ρ is a parameter that controls the persistence of those deviations: ρ closer to 1 means high persistence, ρ closer to 0 implies small persistence.
   Figure 1 depicts TFP evolution with the model parameters calibrated for the US economy: $\hat{\rho} = 0.95, \hat{\sigma}_\varepsilon = 0.007$, $\widehat{\log(1+g)} = 0.005$, and with normalized A_0 to 1. The estimates have been obtained using the US quarterly data from 1948 to the present time.

   Figure 1: Evolution of TFP

[1] You can learn more about autoregressive processes of higher-order and autoregressive moving average models (ARMA) in the textbook (p. 213-214, Remark 63).

2. What are unconditional and conditional expectations of z_t? Show that $\mathbf{E}z_t = 0$ and derive $\mathbf{E}_{t-1}z_t = \rho z_{t-1}$ for all t.

Solution.

Unconditional expectation (expectation about period t formed in period 0):

$$\mathbf{E}z_t = \mathbf{E}[\rho z_{t-1} + \sigma_\varepsilon \varepsilon_t] = \mathbf{E}[\rho(\rho z_{t-2} + \sigma_\varepsilon \varepsilon_{t-1}) + \sigma_\varepsilon \varepsilon_t] = \mathbf{E}[\rho(\rho(\rho z_{t-3} + \sigma_\varepsilon \varepsilon_{t-2}) + \sigma_\varepsilon \varepsilon_{t-1}) + \sigma_\varepsilon \varepsilon_t] =$$
$$= \ldots = \mathbf{E}\left(\rho^{t+1} z_{-1} + \sum_{i=0}^{t} \rho^i \sigma_\varepsilon \varepsilon_{t-i}\right) = \mathbf{E}\left(0 + \sigma_\varepsilon \sum_{i=0}^{t} \rho^i \varepsilon_{t-i}\right) = \sigma_\varepsilon \mathbf{E}\left(\sum_{i=0}^{t} \rho^i \varepsilon_{t-i}\right) =$$
$$= \sigma_\varepsilon \left(\sum_{i=0}^{t} \rho^i \mathbf{E}\varepsilon_{t-i}\right) = \sigma_\varepsilon \left(\sum_{i=0}^{t} (\rho^i \cdot 0)\right) = 0.$$

A faster way to get this result: assume that $\mathbf{E}[z_t] = \mu$ for all t. Then,

$$\mu = \mathbf{E}[\rho z_{t-1} + \sigma_\varepsilon \varepsilon_t] = \rho\mu + \sigma_\varepsilon \cdot 0$$
$$(1-\rho)\mu = 0 \to \mu = 0.$$

Conditional expectation (expectation about period t formed in period $t-1$, implying that the shock of period $t-1$ has been realized and no random anymore):

$$\mathbf{E}_{t-1}z_t = \mathbf{E}_{t-1}[\rho z_{t-1} + \sigma_\varepsilon \varepsilon_t] = \mathbf{E}_{t-1}[\rho z_{t-1}] + \mathbf{E}_{t-1}[\sigma_\varepsilon \varepsilon_t] = \rho z_{t-1} + \sigma_\varepsilon \cdot 0 = \rho z_{t-1}.$$

We can iterate on this procedure to get

$$\mathbf{E}_{t-1} z_{t+j} = \rho^{j+1} z_{t-1},$$

which goes to 0 as $j \to \infty$ since $\rho \in (0,1)$.

3. **(*)** Numerical exercise. Plot the path of the process for technology z_t for 300 periods. Draw 300 productivity shocks ε from the normal distribution $N(0,1)$ and keep them the same for the following scenarios:

- (a) $\rho = 0.1$, $\sigma_\varepsilon = 0.002$
- (b) $\rho = 0.1$, $\sigma_\varepsilon = 0.008$
- (c) $\rho = 0.96$, $\sigma_\varepsilon = 0.002$
- (d) $\rho = 0.96$, $\sigma_\varepsilon = 0.008$.

Solution.

Figure 2 depicts the dynamics of z_t that follows an autoregressive process of order 1 for four cases.

4. How do we measure productivity shocks under the assumptions of the business cycle model with Cobb-Douglas production technology?

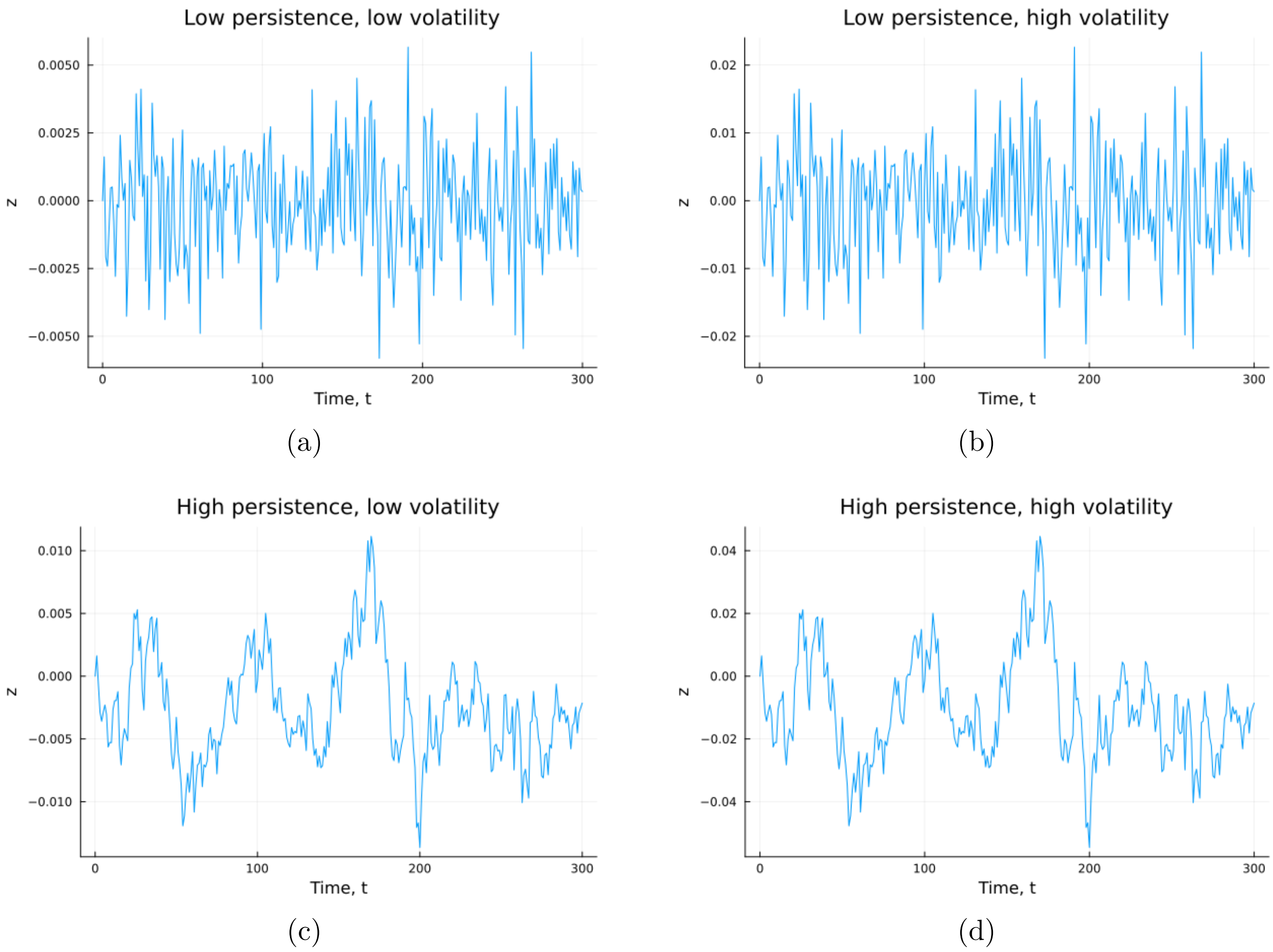
Solution.

1. Using the Cobb-Douglas production function, $y_t = k_t^\alpha (A_t l_t)^{1-\alpha}$ we can express A_t:

$$A_t = \left(\frac{y_t}{k_t^\alpha}\right)^{\frac{1}{1-\alpha}} \frac{1}{l_t}.$$

So, we need to know y_t, k_t, l_t and α to find A_t.

Figure 2: Simulations of four AR(1) processes



2. We can collect a time series for output, y_t, capital k_t, hours worked l_t, and labor income from the national statistical agencies. With the information on labor income share, we can determine α.

3. We find a time series for TFP $\{A_1, ..., A_T\}$ and compute the time trend g and component z_t. We need to estimate a linear regression

$$\log(A_t) = \log(A_0) + [\log(1+g)]t + z_t$$

that is obtained from the assumption on the evolution of A_t:

$$A_t = (1+g)^t A_0 e^{z_t}.$$

3. Using any standard statistical software (or even a spreadsheet) and the series $\{A_1, ..., A_T\}$, either with ordinary least squares (OLS) or maximum likelihood (ML) we get the estimates $\widehat{\log(A_0)}, \widehat{\log(1+g)}$. Part of the output of the regression is the time series of the residuals, which in our case are interpreted as estimated $\hat{z}_t$, $\{\hat{z}_1, ..., \hat{z}_T\}$.

4. Finally, we use $\{\hat{z}_1, ..., \hat{z}_T\}$ to estimate ρ and σ_ε using the following regression:

$$\hat{z}_t = \rho \hat{z}_{t-1} + \sigma_\varepsilon \varepsilon_t.$$

You can review the textbook (FVK), pages 223-225, for more details.