ECON 702 Macroeconomics I

Discussion Handout 8 *

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2 Period Business Cycle

Consider an economy with a representative household and firm across 2 periods $t = 1, 2$. Currently, the economy is stable at $A_1 = 1$. However, in the next period, the economy will be in boom $A_2 = A_H > 1$ with probability p, or bust $A_2 = A_L < 1$ with probability $1 - p$. The household has CRRA utility over consumption in each period, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, exogenously supplies labor $\ell = 1$, is endowed with an initial capital k_0 and can save in physical capital k. The representative competitive firm has Cobb-Douglas production $y_t = A k_t^\alpha \ell_t^{1-\alpha}$.

1. Set up the planner's problem.
2. Derive the Euler equation.
3. Assume $\delta = 1$. Characterize the optimal choice of k.
4. Consider the case when $\sigma = 2$ and when $\sigma = 1$ so that $u(c) = \log(c)$. What happens to the optimal capital decision if households become more optimistic about the future?

Solution:

1.

$$\max_{c_1, c_2, k} \frac{c_1^{1-\sigma}}{1-\sigma} + \beta \left[p \frac{c_{2H}^{1-\sigma}}{1-\sigma} + (1-p) \frac{c_{2L}^{1-\sigma}}{1-\sigma} \right]$$
$$\text{s.t. } c_1 + k = k_0^\alpha + (1-\delta) k_0$$
$$c_{2H} = A_H k^\alpha + (1-\delta) k$$
$$c_{2L} = A_L k^\alpha + (1-\delta) k$$
$$k_0 \text{ given}$$

Note that we have 2 separate budget constraints for each state of the world.

*Teaching Assistants: Anna Lukianova (Email:lukianova@wisc.edu) and John Ryan (Email: john.p.ryan@wisc.edu). Based on the lecture notes by Jesus Fernandez-Villaverde and Dirk Krueger.

2. The Lagrangian is:

$$L = \frac{c_1^{1-\sigma}}{1-\sigma} + \beta\left[p\frac{c_{2H}^{1-\sigma}}{1-\sigma} + (1-p)\frac{c_{2L}^{1-\sigma}}{1-\sigma}\right] +$$
$$\lambda_1\left(k_0^\alpha + (1-\delta)k_0 - c_1 - k\right) +$$
$$\lambda_{2H}\left(A_H k^\alpha + (1-\delta)k - c_{2H}\right) +$$
$$\lambda_{2L}\left(A_L k^\alpha + (1-\delta)k - c_{2L}\right)$$

The first-order conditions are:

$$\begin{aligned} c_1: \quad & c_1^{-\sigma} = \lambda_1 \\ c_{2H}: \quad & \beta p c_{2H}^{-\sigma} = \lambda_{2H} \\ c_{2L}: \quad & \beta(1-p)c_{2L}^{-\sigma} = \lambda_{2L} \\ k: \quad & \lambda_1 = \lambda_{2H}(A_H\alpha k^{\alpha-1} + 1 - \delta) + \lambda_{2L}(A_L\alpha k^{\alpha-1} + 1 - \delta) \end{aligned}$$

Combining the first-order conditions yields:

$$\begin{aligned} c_1^{-\sigma} &= \beta[pc_{2H}^{-\sigma}(\alpha A_H k^{\alpha-1} + 1 - \delta) + (1-p)c_{2L}^{-\sigma}(\alpha A_L k^{\alpha-1} + 1 - \delta)] \\ &= \beta\mathbb{E}[c_2^{-\sigma}(\alpha A_2 k^{\alpha-1} + 1 - \delta)] \end{aligned}$$

3. Plug in the resource / budget constraints into the Euler equation:

$$\begin{aligned} (k_0^\alpha - k)^{-\sigma} &= \alpha\beta k^{\alpha-1}\mathbb{E}[(A_2 k^\alpha)^{-\sigma}A_2] \\ &= \alpha\beta k^{(1-\sigma)\alpha-1}\mathbb{E}[A_2^{1-\sigma}] \\ \implies \frac{k^{1-(1-\sigma)\alpha}}{(k_0^\alpha - k)^\sigma} &= \alpha\beta\mathbb{E}[A_2^{1-\sigma}] \end{aligned}$$

4. If the household becomes more optimistic, we model this as saying p increases.

   The first case is when $\sigma = 2$. Then,

$$\frac{k^{1+\alpha}}{(k_0^\alpha - k)^2} = \alpha\beta\left(\frac{p}{A_H} + \frac{1-p}{A_L}\right)$$

   If p increases, then the right hand side decreases because $\frac{\partial}{\partial p}\left(\frac{p}{A_H} + \frac{1-p}{A_L}\right) = \frac{1}{A_H} - \frac{1}{A_L} < 0$ since $A_H > A_L$.

   The left hand side thus has to decrease as well. But the left hand side is increasing in k, since

$$\frac{\partial}{\partial k}k^{1+\alpha}(k_0^\alpha - k)^{-2} = (1+\alpha)k^\alpha(k_0^\alpha - k)^{-2} + 2k^{1+\alpha}(k_0^\alpha - k)^{-3} > 0$$

So since the left side decreases, it must be that k decreases. This is because in this case, the income effect (need less capital to reach the same level of production) outweighs the substitution effect (capital is more productive).

However, if $\sigma = 1$, then this collapses to

$$\frac{k}{k_0^\alpha - k} = \alpha\beta$$

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We can see that $k = \frac{\alpha\beta k_0^\alpha}{1+\alpha\beta}$, which does not depend on A_2. This is because with log utility (Cobb-Douglas), income and substitution effects perfectly offset each other. Households commit a fixed amount of income to each period, so changes to period in period 2 do not affect period 1 consumption, and thus do not affect the capital choice of the household.

Ricardian Equivalence

Consider a two-period economy where the representative household has preferences:

$$\log(c_1) - \frac{l_1^2}{2} + \beta \left[\log(c_2) - \frac{l_2^2}{2} \right]$$

where c_t is consumption and l_t is labor supply in period t. The household must choose how to allocate its time between leisure and market work in each period. The labor market is perfectly competitive, and the equilibrium wage in each period is denoted w_t.

The production technology uses only labor to produce goods according to:

$$y_t = l_t$$

The government levies a proportional tax on labor income at rate τ_1 in period 1 and τ_2 in period 2. It also imposes lump-sum taxes T_1 and T_2 in periods 1 and 2, respectively. The government uses the tax revenue to finance public goods g in period 1. The government can also issue bonds b at price $1/R$, where R is the gross interest rate.

1. Set up the household's problem and characterize the solution given policy.
2. Derive the government's intertemporal budget constraint, as well as the resource constraint in each period.
3. Solve for the competitive equilibrium in a world where the government doesn't exist.
4. Suppose the government increases g and finances the additional spending through a combination of lump-sum taxes in periods 1 and 2, with no labor taxes. Analyze the effects of this policy on labor supply, consumption, and output in each period. Does the timing of the tax matter for welfare and the allocations in this economy?
5. Now suppose the government finances the increase in g by raising the labor income tax rate in period 1 (τ_1), with no lump sum taxes or labor taxes in period 2. Solve for the competitive equilibrium allocations.
6. Now suppose the government finances the increase in g by raising the labor income tax rate in period 2 (τ_2), with no lump sum taxes or labor taxes in period 1. Solve for the competitive equilibrium allocations.
7. Does the timing of labor taxes matter for welfare or allocations? Why or why not?

Solution:

1. The household's problem is:

$$\max_{c_1,c_2,l_1,l_2,b} \log(c_1) - \frac{l_1^2}{2} + \beta\left[\log(c_2) - \frac{l_2^2}{2}\right]$$

$$\text{s.t.} \quad c_1 + \frac{b}{R} = (1-\tau_1)w_1 l_1 - T_1$$
$$c_2 = (1-\tau_2)w_2 l_2 + b - T_2$$

The first-order conditions are:

$$\frac{1}{c_1} = \lambda_1$$
$$l_1 = \lambda_1(1-\tau_1)w_1$$
$$\frac{\beta}{c_2} = \lambda_2$$
$$l_2 = \lambda_2(1-\tau_2)w_2$$
$$\frac{\lambda_1}{R} = \lambda_2$$

where λ_1 and λ_2 are the Lagrange multipliers associated with the budget constraints.

In equilibrium, $w_1 = w_2 = 1$. Substituting this and combinding first-order conditions:

$$c_1 l_1 = 1 - \tau_1$$
$$c_2 l_2 = 1 - \tau_2$$
$$\frac{c_2}{c_1} = \beta R$$

2. The government's intertemporal budget constraint is obtained by combining its period-by-period budget constraints:

$$g = \tau_1 l_1 + \frac{\tau_2 l_2}{R} + T_1 + \frac{T_2}{R}$$

The resource constraints in each period are:

$$c_1 + g = l_1$$
$$c_2 = l_2$$

3. Without government intervention, the equilibrium allocations are:

$$l_1 = l_2 = 1$$
$$c_1 = c_2 = 1$$
$$R = \frac{1}{\beta}$$

This is clear from combining the resource constraint in each period and the optimality conditions.

4. If the government increases g and finances it through lump-sum taxes, keeping labor income tax rates constant at zero:

   From resource constraint in second period and intratemporal optimality condition:

$$c_2 = l_2 = 1$$
$$l_1 = \frac{g + \sqrt{g^2 + 4}}{2}$$
$$c_1 = \frac{\sqrt{g^2 + 4} - g}{2}$$
$$R = \frac{1}{\beta} \frac{2}{\sqrt{g^2 + 4} - g}$$
$$T_1 + \beta \frac{\sqrt{g^2 + 4} - g}{2} T_2 = g$$

   Output increases in period 1 and remains unchanged in period 2. Private consumption falls in period 1 as g rises. The timing of lump-sum taxes does not matter for the allocations or welfare (Ricardian equivalence).

5. If the government finances the increase in g by raising τ_1, with no lump-sum taxes or second-period labor taxes:

$$c_2 = l_2 = 1$$
$$c_1 = 1 - g$$
$$l_1 = 1$$
$$\tau_1 = g$$
$$R = \frac{1}{\beta(1-g)}$$

   Output does not change in either period, but the interest rate increases to induce lower consumption in period 1.

6. If the government finances the increase in g by raising τ_2, with no lump-sum taxes or first-period labor taxes:

$$l_1 = \frac{g + \sqrt{g^2 + 4}}{2}$$
$$c_1 = \frac{\sqrt{g^2 + 4} - g}{2}$$
$$l_2 = c_2 = \sqrt{1 - \frac{1}{\beta} \frac{2g}{\sqrt{g^2 + 4} - g}} < 1$$
$$\tau_2 = 1 - \sqrt{1 - \frac{1}{\beta} \frac{2g}{\sqrt{g^2 + 4} - g}}$$

   Labor supply increases in period 1 due to the negative wealth effect, but it decreases in period 2 due to the distortionary effect of the labor income tax. Output increases in period 1 and falls in period 2.

7. The timing of labor taxes matters for both welfare and allocations. Taxing labor in period 1 does not distort the labor supply decision in either period, just lower consumption in that period. Taxing labor in period 2 distorts the labor supply decision in both and affects the intertemporal allocation of consumption. We can see that with labor taxes, the equilibrium in each case is strictly worse than the equilibrium with lump sum taxes in terms of consumption. If taxes are in period 1, then consumption is lower in period 1 than the lump sum case and consumption in period 2 is the same. If taxes are in period 2, then consumption is lower in period 2 than the lump sum case and the same in period 1. In general, smoothing labor tax rates over time is more efficient than concentrating the tax burden in a single period.